A Review of Term Structure Estimation Methods

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Introduction

The term structure of interest rates, or the TSIR, can be defined as the relationship between the yield on an investment and the term to maturity of the investment. Many alternative assets such as real estate, private equity, and hedge fund investments are illiquid with long-term cash flows, without a readily available source for market prices. Thus, a properly estimated term structure of interest rates is essential for obtaining the intrinsic values of these assets. Due to the non-linear convex relationship between asset prices and interest rates, any errors in the estimation of interest rates in a low-yield environment have a larger impact on the intrinsic valuation of these assets. Thus, an accurate estimation of the term structure of interest rates assumes even greater importance in the current low-yield environment with a yield around 1% on the short end, and a 3% yield on the 30-year Treasury bond. Moreover, the TSIR is also relevant for macroeconomic forecasts of short-term rates, and implementation of monetary policy and debt policy by governments (see Piazzesi [2010]).

As noted by Bliss [1997], the TSIR estimation requires making three important decisions. First, one must consider the assumptions related to taxes and liquidity premiums in the pricing function that relates bond prices to interest rates or discount factors. Second, one must choose a specific functional form to approximate the interest rates or the discount factors. Moreover, third, one must choose an empirical method for estimating the parameters of the chosen functional form. This paper focuses on how to estimate the default-free term structure of interest rates from bond data using three methods: the bootstrapping method, the McCulloch cubic-spline method, and the Nelson and Siegel method. Nelson and Siegel method is shown to be more robust than the other two methods. The last two methods can be implemented using the user-friendly Excel spreadsheet prepared by the authors.[1]

The structure of the paper is as follows. First, we review the main concepts about the TSIR, such as discount functions, bond prices, yield to maturity, several definitions of interest rates and a discussion of the shape of the TSIR. Next, we describe three popular term structure estimation methods and point out the clues for a proper usage and their limitations.

1. The Building Blocks: Bond Prices, Spot Rates, and Forward Rates

The TSIR can be expressed regarding spot rates, forward rates, or prices of discount bonds. This section shows the relationship between these concepts.

1.1. The Discount Function

Under continuous compounding, the price (or present value) of a zero-coupon bond with a face value of \$100 and a term to maturity of t years can be written as:

$$P(t) = \frac{100}{e^{y(t)t}} = 100\ e^{-y(t)t} = 100\ d(t) \qquad (1)$$

where $y(t)$ is the continuously-compounded rate corresponding to the maturity term t. The function $y(t)$ defines the continuously-compounded term structure based upon zero-coupon rates. The expression $e^{-y(t)t}$ is referred to as the discount function $d(t)$. The typical shape of the discount function is shown in Figure 1. This function starts at 1, since the current value of a \$1 payable today is \$1, and it decreases with increasing maturity due to the time value of money.

If a series of default-free zero-coupon bonds exist for differing maturities, then it is possible to extract the term structure by simply inverting equation (1) to obtain $y(t)$. However, due to the lack of liquidity and unavailability of zero-coupon bonds for all maturities, the term structure cannot be simply obtained by using zero-coupon bonds such as U.S. Treasury STRIPS.

1.2. Bond Price and Accrued Interest

A coupon bond can be viewed as a portfolio of zero-coupon bonds. Using discount function given above, the present value of each coupon paid t_j periods from today is given by $C \times d(t_j)$, where C is the coupon received. This approach can be used to calculate the present value of all the payments, coupons and face value.

This approach gives us P_0, which is called the *cash price* of a bond, and is the price that purchaser pays when buying the bond. However, bond prices are not quoted as cash prices. The quoted prices are *clean prices*, which exclude the accrued interest. Accrued interest is the interest accumulated between the most recent interest payment and the present time. If t_0 denotes the current time, t_p denotes the date of the previous coupon payment, and t_q denotes the date of the next coupon payment, then the formula for accrued interest is given as:

$$AI = C\left(\frac{t_0 - t_p}{t_q - t_p}\right) \qquad (2)$$

and the bond's quoted price is equal to the present value of the all the payments minus the accrued interest. That is,

$$\text{Quoted Price} = P_0 - AI \qquad (3)$$

Computation of accrued interest requires the *day count basis* used in the market. The day count basis defines how to measure the number of days in a year and as well as the number of days between coupons. Note that it is not the cash price, but the quoted price that depends on the specific day count convention being applied. Any increase (decrease) in the accrued interest due to a specific day count convention used is exactly offset by a corresponding decrease (increase) in the quoted price so that the cash price remains unchanged. Since the TSIR is computed using cash prices, it is also independent of the day count convention used. Of course, it is necessary to know the day count convention to obtain the cash price using the quoted price and the accrued interest.

1.3. Yield to Maturity

The yield to maturity is given as that discount rate that makes the sum of the discounted values of all future cash flows (either of coupons or principal) from the bond equal to the cash price of the bond, that is:[2]

$$P = \sum_{j=1}^{N} \frac{C}{e^{y \times t_j}} + \frac{F}{e^{y \times t_N}} \qquad (4)$$

Note that the yield to maturity is a complex weighted average of zero-coupon rates. The size and timing of the coupon payments influence the yield to maturity, and this effect is called the coupon effect. In general, the *coupon effect* will make two bonds with identical maturities but with different coupon rates or payment frequencies have different yields to maturity if the zero-coupon yield curve is non-flat. The coupon effect makes the term structure of yields on coupon bonds lower (higher) than the term structure of zero-coupon rates, when the latter is sloping upward (downward).

1.4. Spot Rates, Forward Rates and Future Rates

Zero-coupon rates as defined above are spot rates because they are interest rates for immediate investments at different maturities. The forward rate between the future dates t_1 and t_2 is the annualized interest rate that can be contractually locked in today on an investment to be made at time t_1 that matures at time t_2. The forward rate is different from the future rate in that the forward rate is known with certainty today, while the future rate can be known only in future.

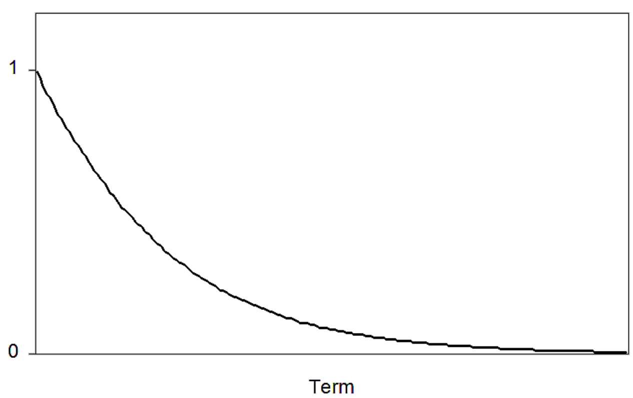

Figure 1: The discount function

Consider two investment strategies. The first strategy requires making a riskless investment of $1 at a future date t_1, which is redeemed at future date t_2 for an amount equal to:

$$1 \times e^{f(t_1,t_2)(t_2-t_1)} \qquad (5)$$

The variable $f(t_1,t_2)$ which is *known today* is defined as the continuously-compounded annualized forward rate, between dates t_1 and t_2.

Now consider a second investment strategy that requires shorting today (which is the same as borrowing and immediately selling) a $1 face value riskless zero-coupon bond that matures at time t_1 and investing the proceeds from the short sale in a riskless investment maturing at time t_2. The proceeds of the short sale equal $P(t_1)$, the current price of $1 face value riskless zero-coupon bond that matures at time t_1. This investment costs nothing today, requires covering the short position at time t_1 by paying $1, and receiving the future value of the proceeds from the short sale. Since both riskless investment strategies require $1 investment at time t_1, and cost nothing today, the value of these investment strategies at time t_2 must be identical. That is, they must offer the same compounded rate of return. This observation can be used to calculate the forward rate that is implied by the term structured observed today. Therefore, the compounded forward rate of return between two future dates t_1 and t_2 is given by:

$$f(t_1,t_2) = y(t_2) + \frac{y(t_2) - y(t_1)}{t_2 - t_1} t_1 \qquad (6)$$

The above equation implies that if the term structure of zero-coupon rates is upward (downward) sloping, then forward rates will be higher (lower) than zero-coupon rates. For a flat term structure, zero-coupon rates and forward rates are identical and equal to a constant.

In general, forward rates can be computed for any arbitrary interval length, and each length implies a different term structure of forward rates. To avoid this indeterminacy, the term structure of forward rates is usually defined using instantaneous forward rates. *Instantaneous forward rates* are obtained when the interval length becomes infinitesimally small.

Mathematically, the instantaneous forward rate $f(t)$, is the annualized rate of return locked-in today, on money to be invested at a future time t, for an infinitesimally small interval. The instantaneous forward rates can be interpreted as the marginal cost of borrowing for an infinitesimal period beginning at time t. By the same token, the annualized time t zero coupon rate can be shown to be equal to the average of all forward rates between now and time t:

$$y(t) \approx \frac{1}{t} \sum_{i=0}^{N} f\left(t_i, t_i + \Delta\right) \qquad (7)$$

The above equation gives a relationship between zero-coupon rates and forward rates. It implies that the zero-coupon rate for term t is an average of the instantaneous forward rates beginning

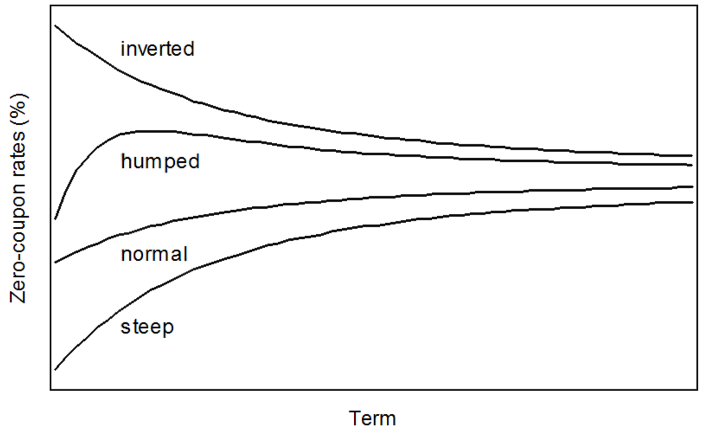

Figure 2: Basic shapes of the term structure

from term 0 to term t. Since averaging reduces volatility, this relationship suggests that forward rates should be in general *more volatile* than zero-coupon rates, especially at the longer end.[3]

1.5. The Shape of the Term Structure of Interest Rates

Estimation of the term structure involves obtaining zero-coupon rates, or forward rates, or discount functions from a set of coupon bond prices. Generally, this requires fitting a functional form that is flexible in capturing stylized facts regarding the shape of the term structure. The TSIR typically takes four different shapes given as the normal shape, the steep shape, the humped shape and the inverted shape. Figure 2 shows these four typical shapes.

The normal shape is indicative of an economy that is normally expanding. That is, the term structure tends to be sloping upwards, reflecting the fact that longer-term investments are riskier. A higher risk implies a higher risk premium and hence, a higher interest rate. The steep shape of the term structure typically occurs at the trough of a business cycle, when after many interest rate reductions by the central bank, the economy seems poised for a recovery in the future. The inverted shape of the term structure typically occurs at the peak of a business cycle, when after many interest rate increases by the central bank, the economic boom or a bubble may be followed by a recession or a depression. Finally, the humped shape typically occurs when the market participants expect a short economic recovery followed by another recession so that there are different expectations at different terms. It could also occur when moving from a normal curve to an inverted curved or vice versa.[4]

It is also worthy to highlight that whatever the shape, the TSIR tend to be horizontal at longest maturities. The reason for this is twofold. First, although investors can hold different expectations about the future of interest rates for the short, medium, and long terms, their long term their expectations are more diffused, which makes it difficult to establish differences between different long rates. Second, risk premiums tend to be more stable for longer terms. This stylized fact should be considered when estimating the TSIR.

2. Three Methods for Term Structure Estimation

First attempts to estimate the term structure relied on fitting smooth functions to the yields to maturity of bonds using regression analysis. However, this approach was unsatisfactory due to its limitation in identifying the zero-coupon yields, and in dealing with the coupon effect. The seminal work of J. Huston McCulloch in 1971 suggested a new method based on quadratic splines, which focused directly on estimating zero-coupon yields and discount factors. Much research has extended the work of McCulloch in the past four decades. Methods for TSIR estimation must find a way to approximate the spot rates, or the forward rates, or the discount function. This requires fitting a parsimonious functional form that is flexible in capturing stylized facts regarding the shape of the term structure. A good term structure estimation method should satisfy the following requirements:

- The method ensures a suitable fitting of the data.
- The estimated zero-coupon rates and the forward rates remain positive over the entire maturity spectrum.
- The estimated discount functions, and the term structures of zero-coupon rates and forward rates are continuous and smooth.
- The method allows asymptotic shapes for the term structures of zero-coupon rates and forward rates at the long end of the maturity spectrum.

The commonly used term structure estimation methods are given as the bootstrapping method, the polynomial/exponential spline methods of McCulloch [1971, 1975] and Vasicek and Fong [1982], and the exponential functional form methods of Nelson and Siegel [1987] and Svensson [1994]. Extensions of the above methods are given as the error weighing models such as the B-spline method of Steely [1991], the penalized spline methods of Fisher, Nychka and Zervos [1995] and Jarrow, Ruppert, and Yu [2004], and the constrained B-spline method of Poletti and Moura [2009], among others.[5] In this paper, we focus on the three most commonly used term structure estimation methods: the bootstrapping method, the McCulloch polynomial cubic-spline method, and the Nelson and Siegel exponential-form method.

2.1. Bootstrapping

The bootstrapping method consists of iteratively extracting zero-coupon yields using a sequence of increasing maturity coupon bond prices.[6] This method requires the existence of at least one bond that matures at each bootstrapping date.

To illustrate this method, consider a set of K bonds that pay semi-annual coupons. The shortest maturity bond is a six-month bond, which by definition does not have any intermediate coupon payments between now and six months, since coupons are paid semi-annually. Using the 6-month zero coupon rate, the price of this bond is given as:

$$P(0.5) = \frac{C_{0.5} + F_{0.5}}{e^{y(0.5)0.5}} \qquad (8)$$

where $F_{0.5}$ is the face value of the bond payable at the maturity of 0.5 years, $C_{0.5}$ is the semi-annual coupon payment at the maturity, and $y(0.5)$ is the annualized six-month zero-coupon yield (under continuously-compounding). The six-month zero-coupon yield can be calculated by taking logarithms of both sides of equation (8), and simplifying as follows:

$$y(0.5) = \frac{1}{0.5}\ln\left[\frac{F_{0.5} + C_{0.5}}{P(0.5)}\right] \qquad (9)$$

In order to compute the 1-year zero-coupon yield, we can use the price of a 1- year coupon bond as follows:

$$P(1) = \frac{C_1}{e^{y(0.5)0.5}} + \frac{F_1 + C_1}{e^{y(1)}} \qquad (10)$$

where F_1 is the face value of the bond payable at the bond's 1-year maturity, C_1 is the semi-annual coupon, which is paid at the end of 0.5 years and 1 year, and $y(1)$ is the annualized 1-year zero-coupon yield. By rearranging the terms in equation (10) and taking logarithms, we get the 1-year zero-coupon yield as follows:

$$y(1) = \ln\left[\frac{F_1 + C_1}{P(1) - \dfrac{C_1}{e^{y(0.5)0.5}}}\right] \qquad (11)$$

Since we already know the six-month yield, $y(0.5)$ from equation (9), this can be substituted in equation (11) to solve for the 1-year yield. Now, continuing in this manner, the six-month yield, $y(0.5)$, and the 1-year yield, $y(1)$, can be both used to obtain the 1.5-year yield, $y(1.5)$, given the price of a 1.5-year maturity coupon bond.

Following the same approach, the zero-coupon yields of all of the K maturities (corresponding to the maturities of the bonds in the sample) are computed iteratively using the zero-coupon yields of the previous maturities.

The zero-coupon yields corresponding to the maturities that lie between these K dates can be computed by using linear or quadratic interpolation. Generally, about 15 to 30 bootstrapping maturities are sufficient in producing the whole term structure of zero-coupon yields. Instead of solving the zero-coupon yields sequentially using an iterative approach as shown above, one can used the matrix approach to solve for all K zero coupon rates simultaneously. Appendix 1 discusses this approach.

The bootstrapping method has two main limitations. First, since this method does not perform optimization, it computes zero-coupon yields that exactly fit the bond prices. This leads to over-fitting since bond prices often contain idiosyncratic errors due to lack of liquidity, bid-ask spreads, special tax effects, etc., and hence, the term structure will not be necessarily smooth as shown in Figure 2. Second, the bootstrapping method requires ad-hoc adjustments when the number of bonds is not the same as the bootstrapping maturities, and when cash flows of different bonds do not fall on the same bootstrapping dates.[7] The next two methods overcome these difficulties by imposing specific functional forms on the term structure.

2.2. Cubic-spline method

Consider the relationship between the *observed* price of a coupon bond maturing at time t_m, and the discount function. As discussed before, the price of this bond can be expressed as the present value of each coupon payment using zero coupon rates:

$$P(t_m) = \sum_{j=1}^{m} CF_j \cdot d(t_j) + \varepsilon \qquad (12)$$

where CF_j is the total cash flow from the bond (i.e., coupon, face value, or both) on date t_j ($j = 1,2,\ldots,m$). Since bond prices are observed with idiosyncratic errors, we need to estimate some functional form for the discount function that minimizes these errors. We face two problems in doing this. First, the discount functions may be highly non-linear, such that we may need a high-dimensional function to make the approximation work. Second, the error terms in equation (12) may increase with the maturity of the bonds, since longer maturity bonds have higher bid-ask spreads, lower liquidity, etc. Due to these, estimation of the discount function using approaches such as least squares minimization, generally fits well at long maturities, but provides a very poor fit at short maturities (see McCulloch [1971] and Chambers Carleton and Waldman [1984]).

The spline method addresses the first issue by dividing the term structure in many segments using a series of points that are called *knotpoints*. Different functions of the same class (polynomial, exponential, etc.) are then used to fit the term structure over these segments. The family of functions is constrained to be continuous and smooth around each knot point to ensure the continuity and smoothness of the fitted curves, using spline methods. McCulloch pioneered the application of splines to term structure estimation by using quadratic polynomial splines in 1971 and cubic polynomial splines in 1975. The cubic spline method remains popular among practitioners and is explained in Appendix 2.

As regard limitations, a potential criticism of the cubic-spline method is the sensitivity of the discount function to the location of the knotpoints. Different knotpoints result in variations in the discount function, which can be sometimes significant. Also, too many knotpoints may lead to overfitting of the discount function. So, one must be careful in the selection of both the number and the placing of the knotpoints.

Another shortcoming of cubic-splines is that they give unreasonably curved shapes for the term structure at the long end of the maturity spectrum, a region where the term structure must have very little curvature. Additionally, the OLS regression used for the estimation of the parameters in equation (26), gives the same weights to the price errors of the bonds with heterogeneous characteristics, such as liquidity, bid-ask spreads, maturity, etc. Other functions can be used for optimization to overcome this limitation but at the cost of precluding the use of OLS techniques.[7]

Finally, the choice of polynomials as basis functions is also controversial. It is argued that the shape of the discount function estimated using cubic splines is usually reasonable up to the maturity of the longest bond in the dataset but tend to be positive or negative infinity when extrapolated to longer terms. This implies that it is possible to generate unbounded positive or negative interest rates. Moreover, although the use of polynomial splines moderates the wavy shape of simple polynomials around the curve to be fitted, this shape might not disappear completely and hence, the fitted discount function might wave around the real discount function introducing a significant variability in both spot and forward rates. Despite these shortcomings, the use of polynomial splines to estimate the TSIR is widespread in the financial industry.

2.3. Nelson and Siegel Model

An alternative approach that overcomes many of the shortcomings of spline techniques is the methodology of Nelson and Siegel. The Nelson and Siegel [1987] model uses a single exponential functional form over the entire maturity range. Nelson and Siegel suggest a parsimonious parameterization of the instantaneous forward rate, which is then used to give a simple representation of the zero coupon curve:

$$y(t) = \alpha_1 + (\alpha_2 + \alpha_3)\frac{\beta}{t}\left(1 - e^{-t/\beta}\right) - \alpha_3 e^{-t/\beta} \qquad (13)$$

The Nelson and Siegel model is based upon four parameters. These parameters can be interpreted as follows:

- $\alpha_1 + \alpha_2$ is the instantaneous short rate, i.e., $\alpha_1 + \alpha_2 = y(0) = f(0)$.
- α_1 is the consol rate. It gives the asymptotic value of the term structure of both the zero-coupon rates and the instantaneous forward rates, i.e., $\alpha_1 = y(\infty) = f(\infty)$.
- The spread between the consol rate and the instantaneous short rate is $-\alpha_2$, which can be interpreted as the slope of the term structure of zero-coupon rates as well as the term structure of forward rates.
- α_3 affects the curvature of the term structure over the intermediate terms. When $\alpha_3 > 0$, the term structure attains a maximum value leading to a concave shape, and when $\alpha_3 < 0$, the term structure attains minimum value leading to a convex shape.
- $\beta > 0$, is the speed of convergence of the term structure towards the consol rate. A lower β value accelerates the convergence of the term structure towards the consol rate, while a higher β value moves the hump in the term structure closer to longer maturities.

Figure 3 illustrates how the parameters α_1, α_2, and α_3, affect the shape of the term structure of zero-coupon rates (given a

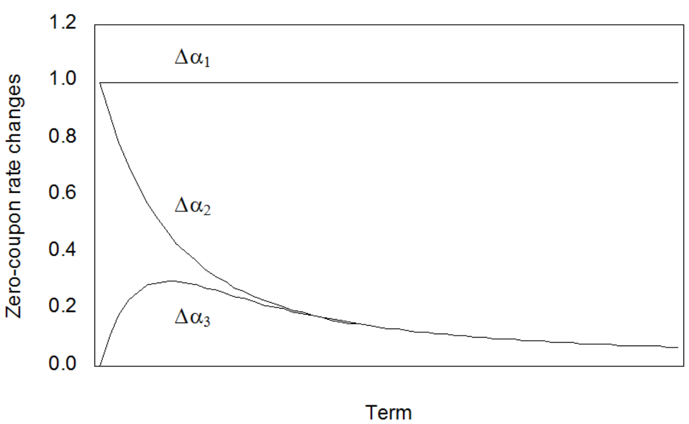

Figure 3: Influence of the alpha parameters of Nelson and Siegel on the term structure of zero-coupon rates

$$\begin{aligned} &\alpha_1 > 0 \\ &\alpha_1 + \alpha_2 > 0 \\ &\beta > 0 \end{aligned} \qquad (15)$$

The first constraint in equation (15) requires that the consol rate remain positive; the second constraint requires that the instantaneous short rate remain positive; finally, the third constraint ensures the convergence of the term structure to the consol rate.

Since the bond pricing equation (14) is a non-linear function, the four parameters are estimated using a non-linear optimization technique. As non-linear optimization techniques are usually sensitive to the starting values of the parameters, these values must be carefully chosen.

Despite this computational difficulty, the Nelson and Siegel model, and its extended version given by Svensson [1994], have a prominent position among term structure estimation methods. The smoothness of the estimated curves for both spot rates and forward rates, the asymptotic behavior of the term structure over the long end, and their robustness to outliers and errors in market data are the main advantages these methods compared to spline methods. In fact, as reported in BIS [2005], most Central Banks use these methods for term structure estimation. Also, in recent years, these models are attracting the interest of researchers in the area of interest modelling and portfolio risk management. Matzner-Løber and Villa [2004] and Diebold and Li [2006], for example, reinterpret them as modern three-factor models of level, slope and curvature factors in the most pure tradition of Litterman and Scheinkman [1991] and Bliss [1997] and obtain empirical evidence in favor of them. Moreover, Christensen, Diebold and Rudebush [2011] provide theoretical foundations for the model by obtaining the affine arbitrage-free dynamic term structure version of the model, which only differs in the existence of a yield-adjustment term, and Krippner [2013] shows that Nelson and Siegel model can be interpreted from the perspective of Gaussian affine term structure models. Finally, Gürkaynak, Sack and Wright [2007] provide the estimates of the US TSIR at a daily frequency from 1961 to present time using the Nelson and Siegel specification for the period before 1980 (due to the lack of long term bonds) and the extension of Svensson [1994] afterwards.

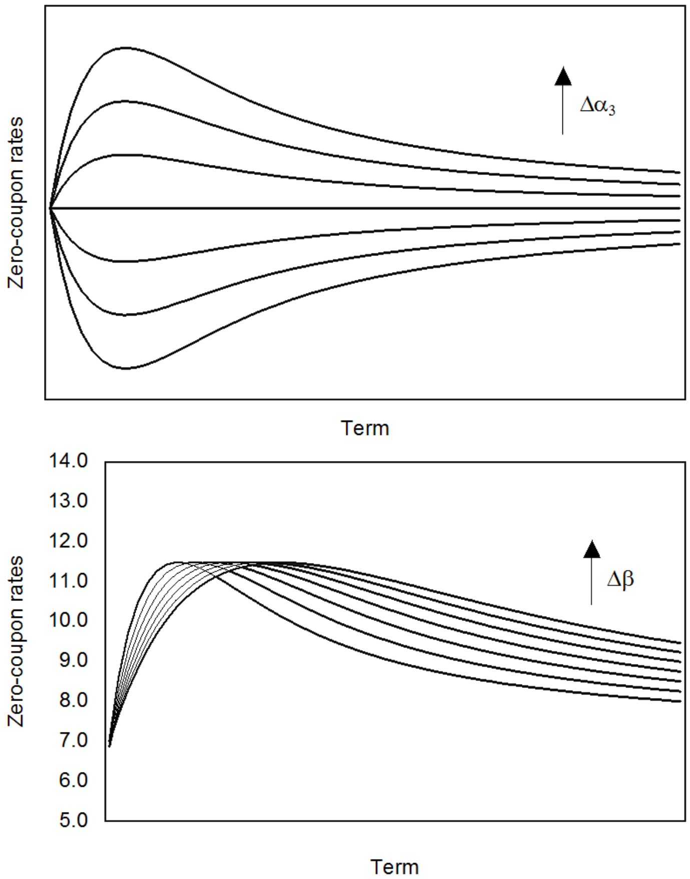

Figure 4: Influence of the curvature and hump positioning parameters of Nelson and Siegel

constant $\beta = 1$). A change in α_1 can be interpreted as the height or parallel change, a change in α_2 can be interpreted as the slope change (though this parameter also affects the curvature change slightly), and a change in α_3 can be interpreted as the curvature change in the term structure of zero-coupon rates.

Figure 4 demonstrates that Nelson and Siegel method is consistent with a variety of term structure shapes, including monotonic and humped, and allows asymptotic behavior of forward and spot rates at the long end. For illustrative purposes, the consol and instantaneous rates have been set at the same level.

The discount function associated with the term structure in (13) can be used to obtain a pricing formula for a coupon-bearing bond, as follows:

$$P(t_m) = \sum_{j=1}^{m} CF_j e^{-\alpha_1 t_j - \beta(\alpha_2+\alpha_3)\left(1-e^{-t_j/\beta}\right) + \alpha_3 t_j e^{-t_j/\beta}} \qquad (14)$$

where t_m is the bond's maturity and CF_j is the cash flow of the bond at time t_j.

The parameters in this equation can be estimated by minimizing the sum of squared errors between the left hand and right hand sides of equation (14) subject to the following constraints:

3. Conclusion

Interest rates play a central role in valuation of financial assets and for making macroeconomic policy. However, they are not directly observable, and should be estimated from the market prices of government securities with different maturities. Many alternative assets such as real estate, private equity, and hedge fund investments are illiquid with long-term cash flows, without a readily available source for market prices. Thus, a properly estimated term structure of interest rates is essential for obtaining the intrinsic values of these assets. In the current low-yield environment, an accurate estimation of the term structure of interest rates assumes even greater importance due to the non-linear convex relationship between asset prices and interest rates. This paper focuses on three commonly used term structure methods, given as the bootstrapping method, the McCulloch cubic spline method and the Nelson and Siegel method. We give a mathematically rigorous illustration, explaining the foundations of the methods, deriving the main equations, and pointing out the advantages and disadvantages of each method.

Appendix 1

The following matrix approach can be used for obtaining a direct solution for the bootstrapping method. Consider *K* bonds maturing at dates t_1, t_2, …, t_K, and let CF_{it} be the total cash flow payments of the ith (for i = 1,2,3,…,*K*) bond on the date t (for $t = t_1, t_2, \ldots, t_K$). Then the prices of the *K* bonds are given by the following system of *K* simultaneous equations:

$$\begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_K) \end{pmatrix} = \begin{pmatrix} CF_{1t_1} & 0 & \cdots & 0 \\ CF_{2t_1} & CF_{2t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CF_{Kt_1} & CF_{Kt_2} & \cdots & CF_{Kt_K} \end{pmatrix} \begin{pmatrix} d(t_1) \\ d(t_2) \\ \vdots \\ d(t_K) \end{pmatrix} \quad (16)$$

Note that the upper triangle of the cash flow matrix on the right-hand side of equation (16) has zero values. By multiplying both sides of equation (16) by the inverse of the cash flow matrix, the discount functions corresponding to maturities $t_1, t_2, \ldots, t_K$ can be computed as follows:

$$\begin{pmatrix} d(t_1) \\ d(t_2) \\ \vdots \\ d(t_K) \end{pmatrix} = \begin{pmatrix} CF_{1t_1} & 0 & \cdots & 0 \\ CF_{2t_1} & CF_{2t_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CF_{Kt_1} & CF_{Kt_2} & \cdots & CF_{Kt_K} \end{pmatrix}^{-1} \begin{pmatrix} P(t_1) \\ P(t_2) \\ \vdots \\ P(t_K) \end{pmatrix} \quad (17)$$

The above solution requires that the number of bonds equals the number of cash flow maturity dates.[8] The zero-coupon rates can be computed from the corresponding discount functions using equation (1).

Appendix 2

Consider a set of *K* bonds with maturities of $t_1, t_2, \ldots, t_K$. years. The range of maturities is divided into *s*-2 intervals defined by *s*-1 knot points $T_1, T_2, \ldots, T_{s-1}$, where $T_1 = 0$ and $T_{s-1} = t_K$. A cubic polynomial spline of the discount function $d(t)$ is defined by the following equation:

$$d(t) = 1 + \sum_{i=1}^{s} \alpha_i g_i(t) \quad (18)$$

where $g_1(t), g_2(t), \ldots, g_s(t)$ define a set of s basis piecewise cubic functions and $\alpha_1, \ldots, \alpha_s$ are unknown parameters that must be estimated.

Since the discount factor for time 0 is 1 by definition, we have:

$$g_i(0) = 0 \qquad i = 1, 2, \ldots, s \quad (19)$$

The continuity and smoothness of the discount function within each interval is ensured by the polynomial functional form of each $g_i(t)$. The continuity and smoothness at the knotpoints is ensured by the requirement that the polynomial functions defined over adjacent intervals (T_{i-1}, T_i) and (T_i, T_{i+1}) have a common value and common first and second derivatives at T_i. The above constraints lead to the following definitions for the set of basis functions $g_1(t), g_2(t), \ldots, g_s(t)$:

Case 1: $i < s$

$$g_i(t) = \begin{cases} 0 & t < T_{i-1} \\ \dfrac{(t - T_{i-1})^3}{6(T_i - T_{i-1})} & T_{i-1} \le t < T_i \\ \dfrac{(T_i - T_{i-1})^2}{6} + \dfrac{(T_i - T_{i-1})(t - T_i)}{2} + \dfrac{(t - T_i)^2}{2} - \dfrac{(t - T_i)^3}{6(T_{i+1} - T_i)} & T_i \le t < T_{i+1} \\ (T_{i+1} - T_{i-1})\left(\dfrac{2T_{i+1} - T_i - T_{i-1}}{6} + \dfrac{t - T_{i+1}}{2}\right) & t \ge T_{i+1} \end{cases} \quad (20)$$

Case 2: $i = s$

$g_i(t) = t$

Substituting equation (18) into equation (12), we can rewrite the price of the bond maturing at date t_m as follows:

$$P(t_m) = \sum_{j=1}^{m} CF_j \left(1 + \sum_{i=1}^{s} \alpha_i g_i(t_j)\right) + \varepsilon \quad (21)$$

By rearranging the terms, we obtain:

$$P(t_m) - \sum_{j=1}^{m} CF_j = \sum_{i=1}^{s} \alpha_i \sum_{j=1}^{m} CF_j g_i(t_j) + \varepsilon \quad (22)$$

The estimation of the discount function requires searching of the unknown parameters, $\alpha_1, \alpha_2, \ldots, \alpha_s$, that minimizes the sum of squared errors across all bonds. Since equation (22) is linear with respect to the parameters $\alpha_1, \alpha_2, \ldots, \alpha_s$, this can be achieved by an ordinary least squares (OLS) regression.

The above approach uses $s-2$ number of maturity segments, $s-1$ number of knotpoints, and s number of cubic polynomial functions. An intuitive choice for the maturity segments may be short-term, intermediate-term, and long-term, which gives three maturity segments of 0 to 1 years, 1 to 5 years, and 5 to 10 years, four knot points given as, 0, 1, 5, and 10 years, and five cubic polynomial functions.

McCulloch recommends choosing knotpoints such that there are approximately equal number of data points (number of bonds' maturities) within each maturity segment. Using this approach, if the bonds are arranged in ascending order of maturity, i.e., $t_1 \le t_2 \le t_3 \ldots \le t_K$, then the knot points are given as follows:

$$T_i = \begin{cases} 0 & i = 1 \\ t_h + \theta(t_{h+1} - t_h) & 2 \le i \le s-2 \\ t_K & i = s-1 \end{cases} \quad (23)$$

where h is an integer defined as:

$$h = INT\left[\frac{(i-1)K}{s-2}\right] \quad (24)$$

and the parameter θ is given as:

$$\theta = \frac{(i-1)K}{s-2} - h \tag{25}$$

McCulloch also suggests that the number of basis functions may be set to the integer nearest to the square root of the number of observations, that is:

$$s = Round\left[\sqrt{K}\right] \tag{26}$$

This choice of s has two desired properties. First, as the number of observations (bonds) increases, the number of basis functions increases. Second, as the number of observations increases, the number of observations within each interval increases, too.

Footnotes

1. The software is available at www.fixedincomerisk.com/web/software.html clicking on the link IRR 1. A Practical Guide to Term Structure Estimation with Excel in the Guides Software section.

2. When compounding is discrete, each exp(yt) is replaced by $(1+y/k)^{tk}$. Since cash price is used in equation (4), sometimes the discount rate is also called the "adjusted" yield to maturity.

3. An excellent visual exposition of the difference in the volatilities of the zero-coupon yields and those of the instantaneous forward rates is given in the excel file TSIRmovie.xls available at www.fixedincomerisk.com/web/software.html clicking on the link Term Structure Movie.

4. The shape of the term structure is also explained by other variables not related to expectations such as liquidity premium, market segmentation, etc. Alternative term structure hypotheses have assigned different roles to these variables. For a brief discussion about the main hypothesis, see Nawalkha, Soto and Beliaeva [2005], pp. 52-55.

5. The method used to estimate the TSIR not only affects these estimates, but also any data derived from them. Diaz, Jareño and Navarro [2011] report this for estimates of interest rate volatility.

6. Usually, not all the bonds that trade in the market at a given time are used for the estimation of the TSIR. The bond selected must cover a wide spectrum of maturities, should have an enough degree of liquidity and their prices shouldn't incorporate high distortions due to tax effects or other market frictions. Usually, these requirements are fulfilled by the establishment of filtering criteria for determining the bonds that qualify for inclusion in the sample.

7. In fact, there are many alternative error-weighing schemes which might lead to more robust estimates of the term structure. For example, Bliss [1997] suggests weighting each bond price error by the inverse of the bond's duration as a way to improve the fitting of long interest rates, which might be poor. This is due to the fact that in absence of a weighting scheme for pricing errors, the quality of the fit of the term structure decreases with maturity. To understand this, consider the relationship between prices, yields and maturities. A same change in price implies a much greater change in yield in short-term bonds compared to long-term bonds. Therefore, following a price error minimization criterion in the estimation will make interest rates corresponding to long-term bonds to be over-fitted at the expense of shorter-term interest rates. Other approaches include the use of penalty functions, as in Fisher, Nychka and Zervos [1995] or Jarrow, Ruppert, and Yu [2004].

8. For example, when two or more bonds mature on the same bootstrapping maturity, the estimated spot rates resulting from using each of these bonds are usually averaged. In the opposite case, when no bond exists at a required bootstrapping maturity, a common practice is to estimate a par yield curve (that is, the yield to maturities of bond priced at par) using simple regression models that make the yields to maturity on current bonds depend on a series of bond characteristics including the coupon rate and the time to maturity. Then, the yields on par bonds are estimated by assuming that the coupon rate of each bond equals its yield to maturity.

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