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Yield Curve Modelling at the Bank of Canada

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ABSTRACT

The primary objective of this paper is to produce a framework that could be used to construct a historical data base of zero-coupon and forward yield curves estimated from Government of Canada securities' prices. The secondary objective is to better understand the behaviour of a class of parametric yield curve models, specifically, the Nelson-Siegel and the Svensson methodologies. These models specify a functional form for the instantaneous forward interest rate, and the user must determine the function parameters that are consistent with market prices for government debt. The results of these models are compared with those of a yield curve model used by the Bank of Canada for the last 15 years. The Bank of Canada's existing model, based on an approach developed by Bell Canada, fits a so-called "par yield" curve to bond yields to maturity and subsequently extracts zero-coupon and "implied forward" rates. Given the pragmatic objectives of this research, the analysis focuses on the practical and deals with two key problems: the estimation problem (the choice of the best yield curve model and the optimization of its parameters); and the data problem (the selection of the appropriate set of market data). In the absence of a developed literature dealing with the practical side of parametric term structure estimation, this paper provides some guidance for those wishing to use parametric models under "real world" constraints.

In the analysis of the estimation problem, the data filtering criteria are held constant (this is the "benchmark" case). Three separate models, two alternative specifications of the objective function, and two global search algorithms are examined. Each of these nine alternatives is summarized in terms of goodness of fit, speed of estimation, and robustness of the results. The best alternative is the Svensson model using a price-error-based, log-likelihood objective function and a global search algorithm that estimates subsets of parameters in stages. This estimation approach is used to consider the data problem. The authors look at a number of alternative data filtering settings, which include a more severe or "tight" setting and an examination of the use of bonds and/or treasury bills to model the short-end of the term structure. Once again, the goodness of fit, robustness, and speed of estimation are used to compare these different filtering possibilities. In the final analysis, it is decided that the benchmark filtering setting offers the most balanced approach to the selection of data for the estimation of the term structure.

This work improves the understanding of this class of parametric models and will be used for the development of a historical data base of estimated term structures. In particular, a number of concerns about these models have been resolved by this analysis. For example, the authors believe that the log-likelihood

specification of the objective function is an efficient approach to solving the estimation problem. In addition, the benchmark data filtering case performs well relative to other possible filtering scenarios. Indeed, this parametric class of models appears to be less sensitive to the data filtering than initially believed. However, some questions remain; specifically, the estimation algorithms could be improved. The authors are concerned that they do not consider enough of the domain of the objective function to determine the optimal set of starting parameters. Finally, although it was decided to employ the Svensson model, there are other functional forms that could be more stable or better describe the underlying data. These two remaining questions suggest that there are certainly more research issues to be explored in this area.

RÉSUMÉ

Le principal objectif des auteurs est d'établir un cadre d'analyse permettant d'élaborer une base de données chronologiques relative aux courbes théoriques de taux de rendement coupon zéro et de taux à terme estimées à partir des cours des titres du gouvernement canadien. Les auteurs cherchent également à mieux comprendre le comportement de la catégorie des modèles paramétriques de courbe de rendement, plus précisément, le modèle de Nelson et Siegel et celui de Svensson. Ces modèles définissent une forme fonctionnelle pour la courbe des taux d'intérêt à terme instantanés, et l'utilisateur doit déterminer les valeurs des paramètres de la fonction qui sont compatibles avec les prix des titres du gouvernement sur le marché. Les résultats obtenus à l'aide de ces modèles sont comparés à ceux du modèle de courbe de rendement que la Banque du Canada utilise depuis quinze ans. Le modèle actuel de la Banque, qui s'inspire d'une approche élaborée par Bell Canada, estime une courbe de « rendement au pair » à partir des taux de rendement à l'échéance des obligations puis en déduit les taux de rendement coupon zéro et les « taux à terme implicites ». Étant donné l'aspect pragmatique des objectifs visés, l'analyse est centrée sur deux importants problèmes d'ordre pratique : le problème de l'estimation (le choix du meilleur modèle pour représenter la courbe de rendement et de la méthode d'optimisation des paramètres) et le problème du choix des données (c'est-à-dire la sélection d'un échantillon approprié parmi les données du marché). Vu l'absence d'une littérature abondante traitant des aspects pratiques de l'estimation de modèles paramétriques relatifs à la structure des taux d'intérêt, les auteurs fournissent quelques conseils à l'intention de ceux qui désirent utiliser les modèles paramétriques dans le cadre des contraintes du « monde réel ».

Pour analyser le problème de l'estimation, les auteurs fixent les critères de filtrage des données (il s'agit de leur « formule de référence » pour le filtrage) et examinent trois modèles distincts, deux spécifications différentes de la fonction objectif et deux algorithmes de recherche globale. Les résultats obtenus à partir de chacun des neuf schémas envisagés sont évalués en fonction de leur robustesse, de l'adéquation statistique et de la vitesse d'estimation. Le schéma qui donne les meilleurs résultats est le modèle de Svensson qui comporte 1) une fonction objectif de type fonction de vraisemblance logarithmique basée sur les erreurs de prix et 2) un algorithme de recherche globale qui estime les sous-ensembles de paramètres par étapes. Les auteurs font ensuite appel à ce schéma d'estimation pour analyser le problème du choix des données. Ils se penchent sur un certain nombre de combinaisons différentes des critères de filtrage des données; ils utilisent un ensemble de critères de filtrage très contraignants d'une part et cherchent à établir

d'autre part si la portion à court terme de la structure des taux est mieux modélisée à l'aide des obligations ou des bons du Trésor (ou des deux types de titres). Les différentes formules de filtrage sont elles aussi comparées entre elles sous l'angle de l'adéquation statistique, de la robustesse et de la vitesse d'estimation. Les auteurs concluent en définitive que la formule de filtrage de référence est la mieux adaptée au choix des données qui serviront à l'estimation de la structure des taux.

Le travail des auteurs contribue à améliorer la compréhension de ce type de modèles paramétriques et permettra d'élaborer une base de données chronologiques relative aux structures de taux estimées. Un certain nombre de questions soulevées par ces modèles ont été résolues dans l'étude. Par exemple, les auteurs croient que la spécification d'une fonction objectif de type fonction de vraisemblance logarithmique est une approche efficace pour résoudre le problème de l'estimation. De plus, la formule de filtrage de référence donne de bons résultats comparativement aux autres formules. Cette catégorie de modèles paramétriques semble en effet moins sensible que prévu au filtrage des données. Toutefois, certaines questions demeurent. En particulier, les algorithmes d'estimation peuvent encore être améliorés. Les auteurs craignent de ne pas avoir couvert une assez grande portion de l'espace de la fonction objectif pour trouver l'ensemble optimal des valeurs de départ des paramètres. En outre, bien qu'ils aient décidé d'utiliser le modèle de Svensson, il se peut que d'autres formes fonctionnelles se révèlent plus stables ou mieux en mesure d'expliquer les données sous-jacentes. Ces deux derniers points laissent croire qu'il subsiste d'autres questions qui méritent d'être explorées dans ce domaine.

1. INTRODUCTION

Zero-coupon and forward interest rates are among the most fundamental tools in finance. Applications of zero-coupon and forward curves include measuring and understanding market expectations to aid in the implementation of monetary policy; testing theories of the term structure of interest rates; pricing of securities; and the identification of differences in the theoretical value of securities relative to their market value. Unfortunately, zero-coupon and forward rates are not directly observable in the market for a wide range of maturities. They must, therefore, be estimated from existing bond prices or yields.

A number of estimation methodologies exist to derive the zero-coupon and forward curves from observed data. Each technique, however, can provide surprisingly different shapes for these curves. As a result, the selection of a specific estimation technique depends on its final use. The main interest of this paper in the term structure of interest rates relates to how it may be used to provide insights into market expectations regarding future interest rates and inflation. Given that this application does not require pricing transactions, some accuracy in the "goodness of fit" can be foregone for a more parsimonious and easily interpretable form. It is nevertheless important that the estimated forward and zero-coupon curves fit the data well.

The primary objective of this paper is to produce a framework that could be used to generate a historical data base of zero-coupon and forward curves estimated from Government of Canada securities' prices. The purpose of this research is also to better understand the behaviour of a different class of yield curve model in the context of Canadian data. To meet these objectives, this paper revisits the Bank of Canada's current methodology for estimating Canadian government zero-coupon and forward curves. It introduces and compares this methodology with alternative approaches to term structure modelling that rely upon a class of parametric models, specifically, the Nelson-Siegel and the Svensson methodologies.

The Bank's current approach utilizes the so-called Super-Bell model for extracting the zero-coupon and forward interest rates from Government of Canada bond yields. This approach uses essentially an ordinary least-squares (OLS) regression to fit a par yield curve from existing bond "yields to maturity" (YTM). It then employs a technique termed "bootstrapping" to derive zero-coupon rates and subsequently implied forward rates. The proposed models are quite different from the current approach and begin with a specified parametrized functional form for the instantaneous forward rate curve. From this functional form, described later in the text, a continuous zero-coupon rate function and its respective discount function are derived. An optimization process is used to determine the appropriate parameters for these functions that best fit the existing bond prices.

The research has pragmatic objectives, so the focus throughout the analysis is highly practical. It deals with two key problems: the *estimation problem*, or the choice of the best yield curve model and the optimization of its parameters; and the *data problem*, or the selection of the appropriate set of market data. The wide range of possible filtering combinations and estimation approaches makes this a rather overwhelming task. Therefore, examination is limited to a few principal dimensions. Specifically, the analysis begins with the definition of a "benchmark" filtering case. Using this benchmark, the estimation problem is examined by analyzing different objective function specifications and optimization algorithms. After this analysis, the best optimization approach is selected and used to consider two different aspects of data filtering. To accomplish this, different data filtering scenarios are contrasted with the initial benchmark case.

Section 2 of the paper introduces the current Super-Bell model and the proposed Nelson-Siegel and Svensson models and includes a comparison of the two modelling approaches. Section 3 follows with a description of Canada bond and treasury bill data. This section also details the two primary data filtering dimensions: the severity of data filtering, and the selection of observations at the short-end of the maturity spectrum. The empirical results, presented in Section 4, begin with the treatment of the estimation problem followed by the data problem. The final section, Section 5, presents some concluding remarks.

2. THE MODELS

The following section details how the specific yield curve models selected are used to extract theoretical zero-coupon and forward interest rates from observed bond and treasury bill prices. The new yield curve modelling methodology introduced in this section is fundamentally different from the current Super-Bell model. To highlight these differences, the current methodology is discussed briefly and then followed by a detailed description of the new approach. The advantages and disadvantages of each approach are also briefly detailed.

2.1 The Super-Bell model

The Super-Bell model, developed by Bell Canada Limited in the 1960s, is quite straightforward. It uses an OLS regression of yields to maturity on a series of variables including power transformations of the term to maturity and two coupon terms. The intent is to derive a so-called par yield curve.[1] A par yield curve is a series of yields that would be observed if the sample of bonds were all trading at par value. The regression equation is as follows:

$$Y_{M,C} = \beta_0 + \beta_1(M) + \beta_2(M^2) + \beta_3(M^3) + \beta_4(M^{0.5}) + \beta_5(\log M) + \beta_6(C) + \beta_7(C \cdot M) + \varepsilon \qquad \text{(EQ 1)}$$

1. See Section A.4, "Par yields," on page 42 in the technical appendix for a complete definition of par yields.

This regression defines yield to maturity ($Y_{M,C}$) as a function of term to maturity (M) and the coupon rate (C). Once the coefficients (β_0 through β_6) have been estimated, another regression is performed to extract the par yields. By definition, a bond trading at par has a coupon that is equal to the yield (that is, $Y_{M,C} = C$). As a result, the expression above can be rewritten as follows:

$$Y_M = \frac{\beta_0 + \beta_1(M) + \beta_2(M^2) + \beta_3(M^3) + \beta_4(M^{0.5}) + \beta_5(\log M)}{1 - \beta_6 + \beta_7(M)} + \varepsilon \quad \text{(EQ 2)}$$

Using the coefficients estimated from the first equation and the term to maturity for each bond, a vector of par yields (Y_M) is obtained through this algebraic rearrangement of the original regression equation. The second step uses this vector of par yields and runs an additional estimation, using the same term-to-maturity variables but without the coupon variables as follows to create a "smoothed" par yield curve:

$$Y_M = \beta_0 + \beta_1(M) + \beta_2(M^2) + \beta_3(M^3) + \beta_4(M^{0.5}) + \beta_5(\log M) + \varepsilon \quad \text{(EQ 3)}$$

In 1987, however, an adjustment was made to the par yield estimation. Specifically, a different estimation is used to obtain a par yield vector for bonds with a term to maturity of 15 years and greater. The following specification is used, making the explicit assumption that the term to maturity is a linear function of the coupon rate.[2] The impact of this approach, which makes the coupon effect constant for all bonds with terms to maturity of 15 years and greater, is to flatten out the long end of the yield curve.

$$Y_{M>15,\,C} = \beta_0 + \beta_1(C) + \varepsilon \quad \text{(EQ 4)}$$

The par yield values for longer-term bonds are therefore solved using the same assumption of $Y_{M,C} = C$, as follows:

$$Y_{M>15} = \frac{\beta_0}{1 - \beta_1} \quad \text{(EQ 5)}$$

The par yield values are combined for all maturities and the new par yield curve is estimated using the same approach as specified above in equation (2). From these estimated coefficients, the corresponding theoretical par yields can be obtained for any set of maturities.

2. This is unlike the specification for yields with a term to maturity of less than 15 years where the coupon effect is permitted to take a non-linear form.

The determination of the par yield curve is only the first step in calculating zero-coupon and forward interest rates. The next step is to extract the zero-coupon rates from the constant maturity par yield curve, using a technique termed “bootstrapping.” Bootstrapping provides zero-coupon rates for a series of discrete maturities. In the final step, the theoretical zero-coupon rate curve is used to calculate implied forward rates for the same periodicity. Implied forward rate calculation and bootstrapping are described in the Technical Appendix of this paper.

Advantages of the Super-Bell model, which dates back more than 25 years, include the following:

- The model is not conceptually difficult.
- The model is parametrized analytically and is thus straightforward to solve.

There are, however, several criticisms of the Super-Bell model:

- The resulting forward curve is a by-product of a lengthy process rather than the primary output of the Super-Bell model.
- The Super-Bell model focuses exclusively on YTM rather than on the actual cash flows of the underlying bonds.
- The zero-coupon curve can be derived only for discrete points in time. It is, therefore, necessary to make additional assumptions to interpolate between the discrete zero-coupon rates.

As a consequence of these shortcomings, the Super-Bell model can lead to forward curves with very strange shapes (particularly at longer maturities) and poor fit of the underlying bond prices or yields.

2.2 The Nelson-Siegel and Svensson models

The basic parametric model presented in this paper was developed by Charles Nelson and Andrew Siegel of the University of Washington in 1987. The Svensson model is an extension of this previous methodology.[3] Since the logic underlying the models is identical, the text will focus on the more sophisticated Svensson model.

3. As a result, the Svensson model is often termed the extended Nelson and Siegel model. This terminology is avoided in the current paper because other possible extensions to the base Nelson and Siegel model exist. See Nelson and Siegel (1987) and Svensson (1994).

Continuous interest rate concepts are critically important to any understanding of the Nelson-Siegel and Svensson methodologies. Consequently, these concepts will be briefly introduced prior to the models being described. In general practice, interest rates are compounded at discrete intervals. In order to construct continuous interest rate functions (i.e., a zero-coupon or forward interest rate curve), the compounding frequency must also be made continuous. It should be noted, however, that the impact on zero-coupon and forward rates due to the change from semi-annual to continuous compounding is not dramatic.[4]

On a continuously compounded basis, the zero-coupon rate $z(t,T)$ can be expressed as a function of the discretely compounded zero-coupon rate $Z(t,T)$ and the term to maturity, T, as follows:

$$z(t, T) = e^{\left(\frac{Z(t, T)}{100} \cdot (T - t)/365\right)} \quad \text{(EQ 6)}$$

The continuously compounded discount factor can be similarly expressed:

$$disc(t, T) = e^{-\left(\frac{Z(t, T)}{100} \cdot (T - t)/365\right)} \quad \text{(EQ 7)}$$

The forward rate can also be represented as a continuously compounded rate:

$$f(t, \tau, T) = \frac{[(T - t) \cdot z(t, T)] - [(\tau - t) \cdot z(t, \tau)]}{T - \tau} \quad \text{(EQ 8)}$$

Another important concept is the instantaneous forward rate ($f(t, \tau, T)_{INST}$). This is the limit of the previous expression (shown in equation 8) as the term to maturity of the forward contract tends towards zero:

$$f(t, \tau, T)_{INST} = \lim_{\tau \to T} f(t, \tau, T) \quad \text{(EQ 9)}$$

The instantaneous forward rate can be defined as the marginal cost of borrowing (or marginal revenue from lending) for an infinitely short period of time. In practice, it would be equivalent to a forward overnight interest rate. The continuously compounded zero-coupon rate for the same period of time, $z(t,T)$, is the average cost of borrowing over this period. More precisely, the zero-coupon rate at time t with maturity T is equal to the average of the instantaneous forward rates with trade dates between time t and T. The standard relationship between marginal and

4. For example, a 10-year zero-coupon bond discounted with a 10 per cent, 10-year annually compounded zero-coupon rate has a price of $38.54. The same zero-coupon bond discounted with a 10 per cent, 10-year continuously compounded zero-coupon rate has a price of $36.79.

average cost can be shown to hold between forward rates (marginal cost) and zero-coupon rates (average cost); that is, the instantaneous forward rate is the first derivative of the zero-coupon rate with respect to term to maturity. Thus, if equation 6 is differentiated with respect to time, the following expression will be obtained:

$$f(t, \tau, T)_{INST} = z(t, T) + (T - t) \cdot \frac{\partial z(t, T)}{\partial t} \quad \text{(EQ 10)}$$

Equivalently, the zero-coupon rate is the integral of the instantaneous forward rate in the interval from settlement (time *t*) to maturity (time *T*), divided by the number of periods to determine a period zero-coupon rate. It is summarized as follows:

$$z(t, T) = \frac{\int_{x=t}^{T} f(t, \tau, T)_{INST} dx}{T - t} \quad \text{(EQ 11)}$$

This important relationship between zero-coupon and instantaneous forward rates is a critical component of the Nelson-Siegel and Svensson models.

The Svensson model is a parametric model that specifies a functional form for the instantaneous forward rate, *f(TTM)*, which is a function of the term to maturity *(TTM)*. The functional form is as follows:[5]

$$f(TTM)_t = \beta_0 + \beta_1 \left(e^{-\left(\frac{TTM}{\tau_1}\right)} \right) + \beta_2 \left[\frac{TTM}{\tau_1} \left(e^{-\left(\frac{TTM}{\tau_1}\right)} \right) \right] + \beta_3 \left[\frac{TTM}{\tau_2} \left(e^{-\left(\frac{TTM}{\tau_2}\right)} \right) \right] \quad \text{(EQ 12)}$$

The original motivation for this modelling method was a desire to create a parsimonious model of the forward interest rate curve that could capture the range of shapes generally seen in yield curves: a monotonic form, humps at various areas of the curve, and s-shapes. This is one possibility among numerous potential functional forms that could be used to fit a term structure. The Svensson model is a good choice, given its ability to capture the stylized facts describing the behaviour of the forward curve.[6]

This model has six parameters that must be estimated, β_0, β_1, β_2, β_3, τ_1, and τ_2. As illustrated in Figure 1, these parameters identify four different curves, an asymptotic value, the general

5. $f(TTM)_t$ is the functional equivalent of $f(t,\tau,T)_{INST}$ with $(\tau\text{-}t) \rightarrow (T\text{-}t) = TTM$.

6. Note, however, that this approach is essentially an exercise in curve fitting, guided by stylized facts, and is not directed by any economic theory.

shape of the curve, and two humps or U-shapes, which are combined to produce the Svensson instantaneous forward curve for a given date. The impact of these parameters on the shape of the forward curve can be described as follows:[7]

- β_0 = This parameter, which must be positive, is the asymptotic value of $f(TTM)_t$. The curve will tend towards the asymptote as the *TTM* approaches infinity.

- β_1 = This parameter determines the starting (or short-term) value of the curve in terms of deviation from the asymptote. It also defines the basic speed with which the curve tends towards its long-term trend. The curve will have a negative slope if this parameter is positive and vice versa. Note that the sum of β_0 and β_1 is the vertical intercept.

- τ_1 = This parameter, which must also be positive, specifies the position of the first hump or U-shape on the curve.

- β_2 = This parameter determines the magnitude and direction of the hump. If β_2 is positive, a hump will occur at τ_1 whereas, if β_2 is negative, a U-shaped value will occur at τ_1.

- τ_2 = This parameter, which must also be positive, specifies the position of the second hump or U-shape on the curve.

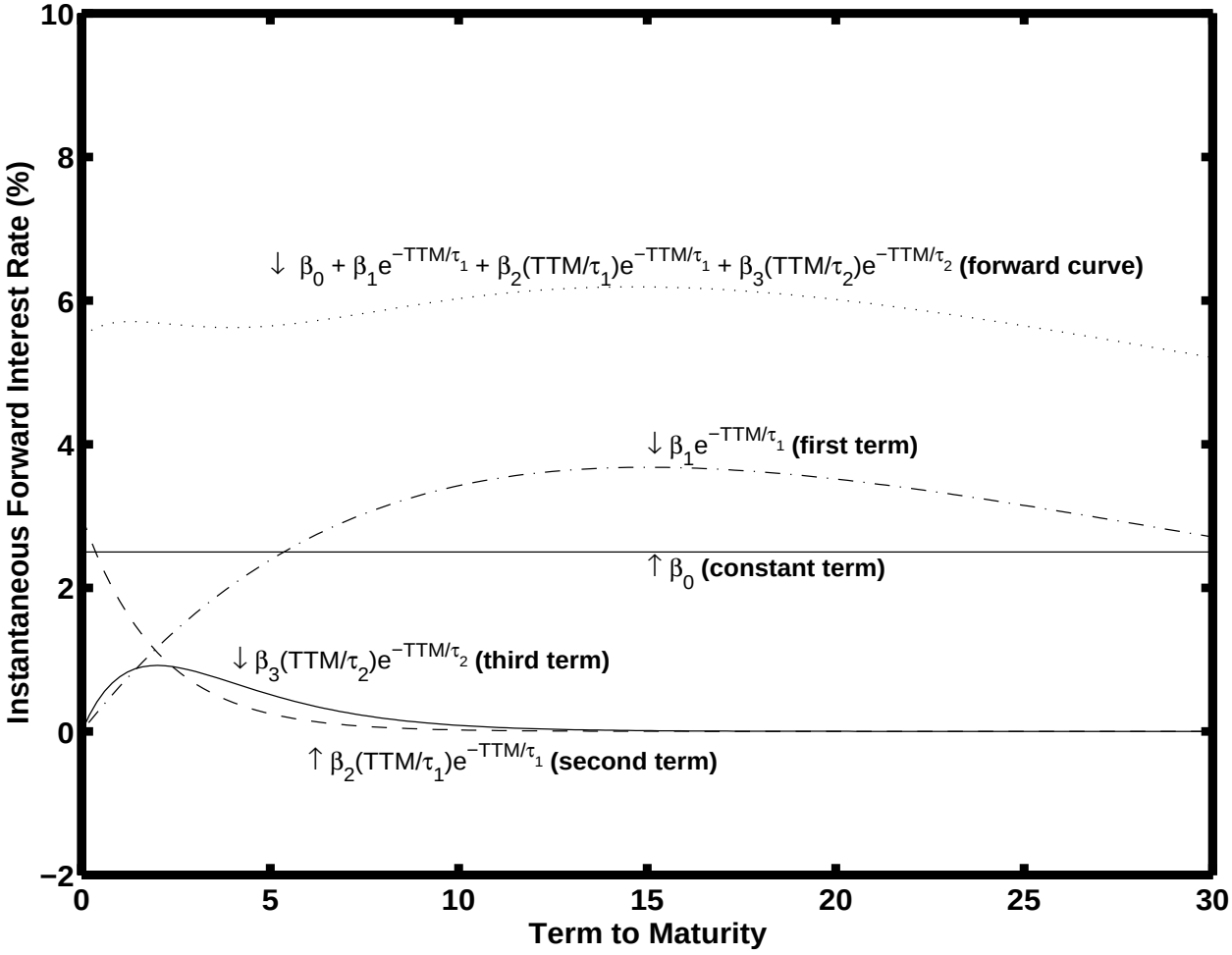
- β_3 = This parameter, in a manner analogous to β_2, determines the magnitude and direction of the second hump.

7. The difference between the Nelson-Siegel (one-hump) and Svensson (two-hump) versions of the model is the functional form of the forward curve. In the one-hump version, the forward curve is defined as follows:

$$f(TTM)_t = \beta_0 + \beta_1\left(e^{-\left(\frac{TTM}{\tau_1}\right)}\right) + \beta_2\left[\frac{TTM}{\tau_1}\left(e^{-\left(\frac{TTM}{\tau_1}\right)}\right)\right]$$

As a result, this model has only four parameters that require estimation; the β_3 and τ_2 parameters do not exist in this model (i.e., β_2 and τ_2 equal zero in the Nelson-Siegel model).

Figure 1. A decomposition of the forward term structure functional form



Having specified a functional form for the instantaneous forward rate, a zero-coupon interest rate function is derived. This is accomplished by integrating the forward function. As previously discussed, this is possible, given that the instantaneous forward rate (which is simply the marginal cost of borrowing) is the first derivative of the zero-coupon rate (which is similarly the average cost of borrowing over some interval). This function is summarized as follows:

$$z(TTM)_t = \beta_0 + \beta_1\left[\frac{1-e^{\left(\frac{-TTM}{\tau_1}\right)}}{(TTM)/\tau_1}\right] + \beta_2\left(\frac{1-e^{\left(\frac{-TTM}{\tau_1}\right)}}{(TTM)/\tau_1} - e^{\left(\frac{-TTM}{\tau_1}\right)}\right) + \beta_3\left(\frac{1-e^{\left(\frac{-TTM}{\tau_2}\right)}}{(TTM)/\tau_2} - e^{\left(\frac{-TTM}{\tau_2}\right)}\right) \quad \text{(EQ 13)}$$

It is then relatively straightforward to determine the discount function from the zero-coupon function.

$$disc(TTM)_t = e^{-\left(\frac{z(TTM)_t}{100} \cdot (TTM)\right)} \quad \text{(EQ 14)}$$

Once the functional form is specified for the forward rate, it permits the determination of the zero-coupon function and finally provides a discount function. The discount function permits the discounting of any cash flow occurring throughout the term-to-maturity spectrum.

The instantaneous forward rate, zero-coupon, and discount factor functions are closely related, with the same relationship to the six parameters. The zero-coupon and discount factor functions are merely transformations of the original instantaneous forward rate function. The discount function is the vehicle used to determine the price of a set of bonds because the present value of a cash flow is calculated by taking the product of this cash flow and its corresponding discount factor. The application of the discount factor function to all the coupon and principal payments that comprise a bond provides an estimate of the price of the bond. The discount factor function, therefore, is the critical element of the model that links the instantaneous forward rate and bond prices.

Every different set of parameter values in the discount rate function (which are equivalently different in the zero-coupon and instantaneous forward rate functions) translates into different discount factors and thus different theoretical bond prices. What is required is to determine those parameter values that are most consistent with observed bond prices. The basic process of determining the optimal parameters for the original forward function that best fit the bond data is outlined as follows:[8]

A. A vector of starting parameters [β_0, β_1, β_2, β_3, τ_1, τ_2] is selected.

B. The instantaneous forward rate, zero-coupon, and discount factor functions are determined, using these starting parameters.

C. The discount factor function is used to determine the present value of the bond cash flows and thereby to determine a vector of theoretical bond prices.

D. Price errors are calculated by taking the difference between the theoretical and observed prices.

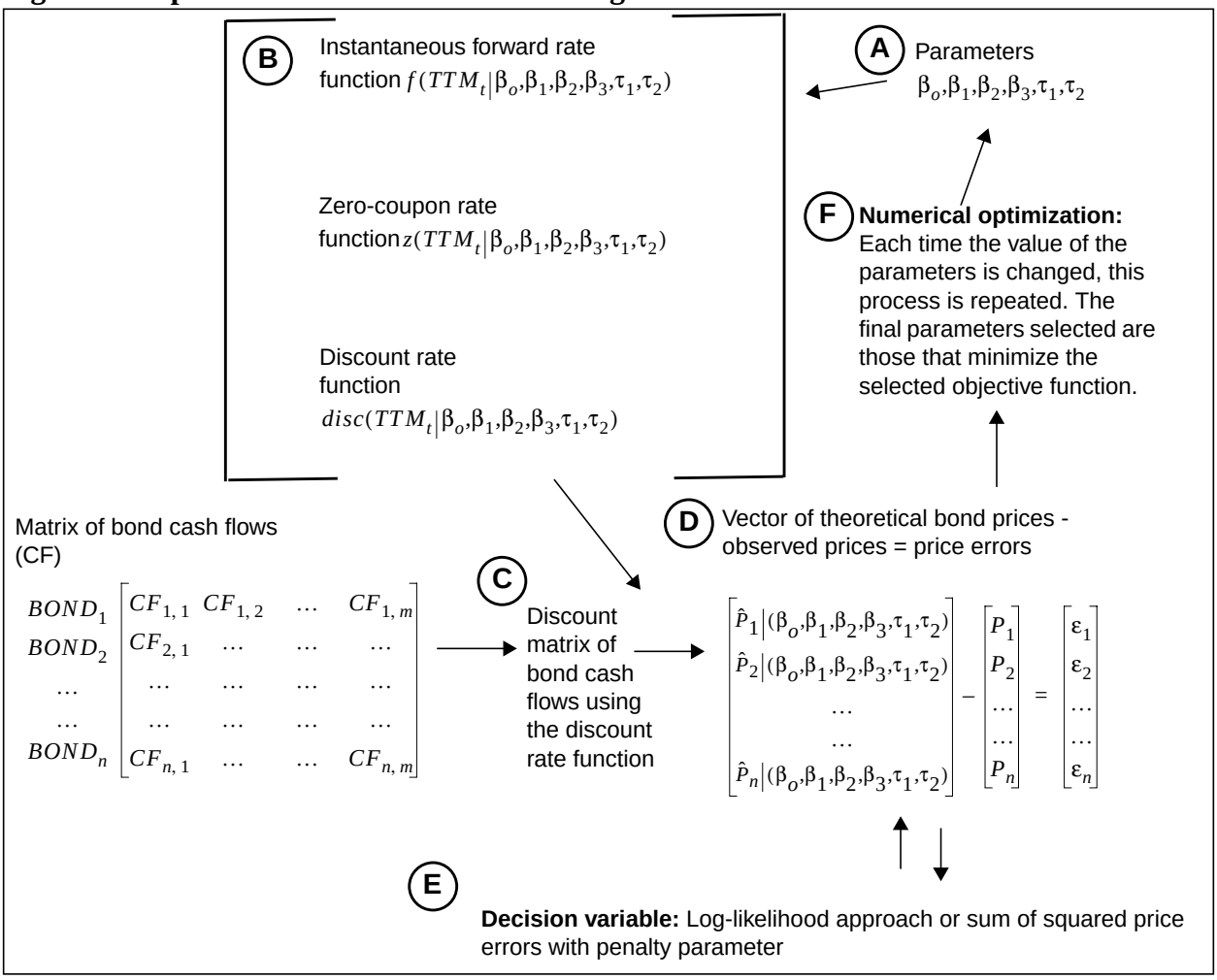
8. For more details of this process, see Technical Appendix, Section D, “Mechanics of the estimation,” on page 46.

E. Two different numerical optimization procedures, discussed later in greater detail, are used to minimize the decision variable subject to certain constraints on the parameter values.[9]

F. Steps B through E are repeated until the objective function is minimized.

The final parameter estimates are used to determine and plot the desired zero-coupon and forward interest rate values. Figure 2 details the previously outlined process following the steps from A to F.

Figure 2. Steps in the estimation of Nelson-Siegel and Svensson models



As indicated, the process describes the minimization of price errors rather than YTM errors. Price errors were selected because the yield calculations necessary to minimize YTM errors are prohibitively time consuming in an iterative optimization framework. In contrast to the

9. In particular, the β_0 and τ_1 values are forced to take positive values and the humps are restricted to fall between 0 and 30 years, which corresponds to the estimation range.

calculation of the bond price, each YTM calculation relies on a time-consuming numerical approximation procedure.[10]

It is nevertheless critical that the model be capable of consistently fitting the YTMs of the full sample of bonds used in the estimation. The intention is to model the yield curve, not the price curve. Focusing on price errors to obtain YTMs can create difficulties. As a result, an important element in the optimization is the weighting of price errors, a procedure necessary to correct for the heteroskedasticity that occurs in the price errors. To understand how this is problematic, one needs to consider the relationship between yield, price, and the term to maturity of a bond. This relationship is best explained by the concept of duration.[11] A given change in yield leads to a much smaller change in the price of a 1-year treasury bill than a 30-year long bond. The corollary of this statement is that a large price change for a 30-year long bond may lead to an identical change in yield when compared to a much smaller price change in a 1-year treasury bill. The optimization technique that seeks to minimize price errors will therefore tend to try to reduce the heteroskedastic nature of the errors by overfitting the long-term bond prices at the expense of the short-term prices. This in turn leads to the overfitting of long-term yields relative to short-term yields and a consequent underfitting of the short-end of the curve. In order to correct for this problem, each price error is simply weighted by a value related to the inverse of its duration. The general weighting for each individual bond has the following form:[12]

$$\omega_i = \frac{1/D_i}{\sum_{j=1}^{n} 1/D_j} \quad \text{where } D_i \text{ is the MacCauley duration of the } ith \text{ bond.}[13] \qquad \text{(EQ 15)}$$

There are a number of advantages of the Nelson-Siegel and Svensson approach compared with the Super-Bell model:

- The primary output of this model is a forward curve, which can be used as an approximation of aggregate expectations for future interest rate movements.

10. Moreover, the time required for the calculation is an increasing function of the term to maturity of the underlying bond. In addition, the standard Canadian yield calculations, particularly as they relate to accrued interest, are somewhat complicated and would require additional programming that would serve only to lengthen the price-to-yield calculation.
11. See Technical Appendix, Section A.3 on page 42 for more on the concept of duration.
12. The specifics of the weighting function are described in the Technical Appendix, Section B, on page 43. Note that this general case has been expanded to also permit altering the weighting of benchmark bond and/or treasury bill price errors.
13. This is consistent with the Bliss (1991) approach.

- This class of models focuses on the actual cash flows of the underlying securities rather than using the yield-to-maturity measure, which is subject to a number of shortcomings.[14]

- The functional form of the Nelson-Siegel and Svensson models was created to be capable of handling a variety of the shapes that are observed in the market.

- These models provide continuous forward, zero-coupon, and discount rate functions, which dramatically increase the ease with which cash flows can be discounted. They also avoid the need to introduce other models for the interpolation between intermediate points.

Nevertheless, there are some criticisms of this class of term structure model. Firstly, there is a general consensus that the parsimonious nature of these yield curve models, while useful for gaining a sense of expectations, may not be particularly accurate for pricing securities.[15]

The main criticism of the Nelson-Siegel and Svensson methodologies, however, is that their parameters are more difficult to estimate relative to the Super-Bell model. These estimation difficulties stem from a function that, while linear in the beta parameters, is non-linear in the taus. Moreover, there appear to be multiple local minima (or maxima) in addition to a global minimum (or global maximum). To attempt to obtain the global minimum, it is therefore necessary to estimate the model for many different sets of starting values for the model parameters. Complete certainty on the results would require consideration of virtually all sets of starting values over the domain of the function; this is a very large undertaking considering the number of parameters. With six parameters, all possible combinations of three different starting parameter values amount to $3^6 = 729$ different starting values, while five different starting values translate into $5^6 = 15,625$ different sets of starting values. The time required to estimate the model, therefore, acts as a constraint on the size of the grid that could be considered and hence the degree of precision that any estimation procedure can attain.

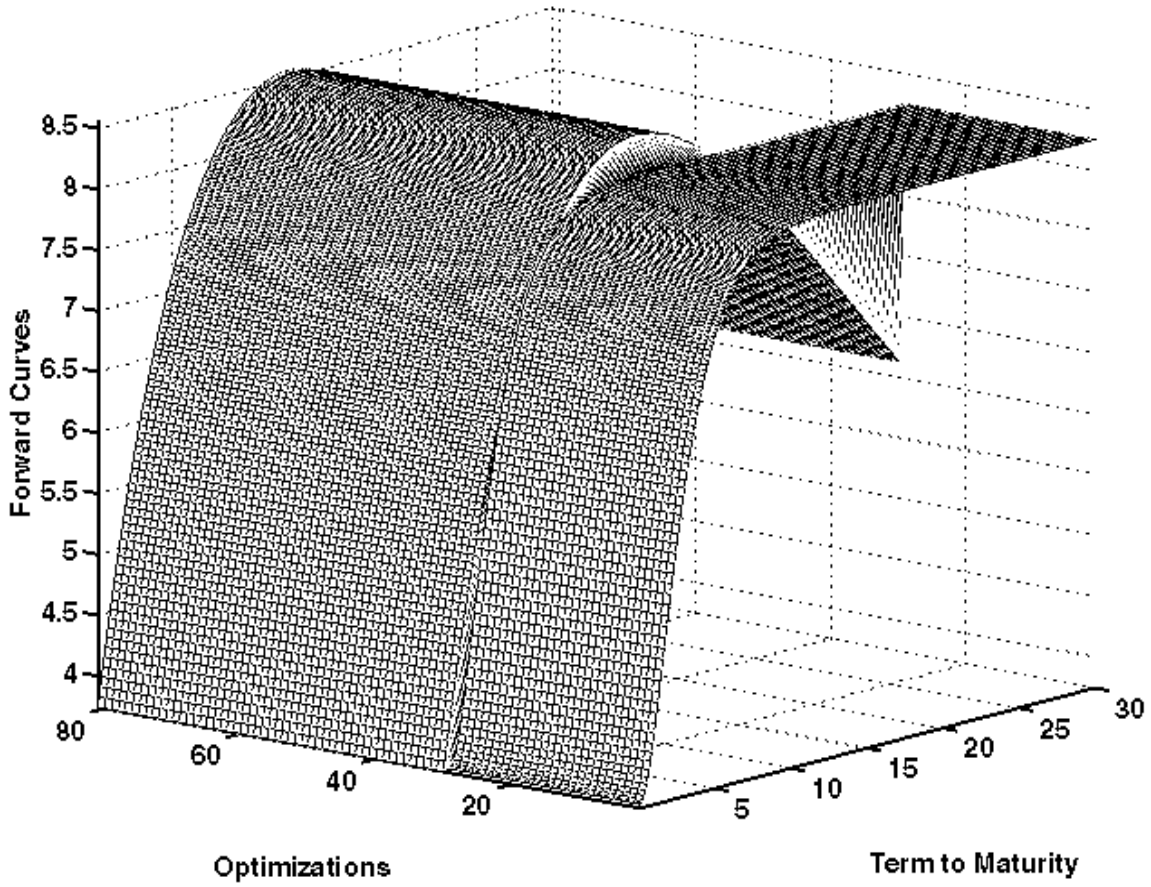
By way of example, Figures 3 and 4 demonstrate the sensitivity of the Nelson-Siegel and Svensson models to the starting parameter values used for a more dramatic date in the sample, 17 January 1994. In Figure 3, only 29 of the 81 sets of starting values of the parameters for the Nelson-Siegel model on that date give a forward curve close to the best one estimated within this

14. See Technical Appendix, Section A.2, “Yield to maturity and the ‘coupon effect’,” page 40 for more details.

15. This is because, by the very nature as a “parsimonious” representation of the term structure, they fit the data less accurately than some alternative models such as cubic splines.

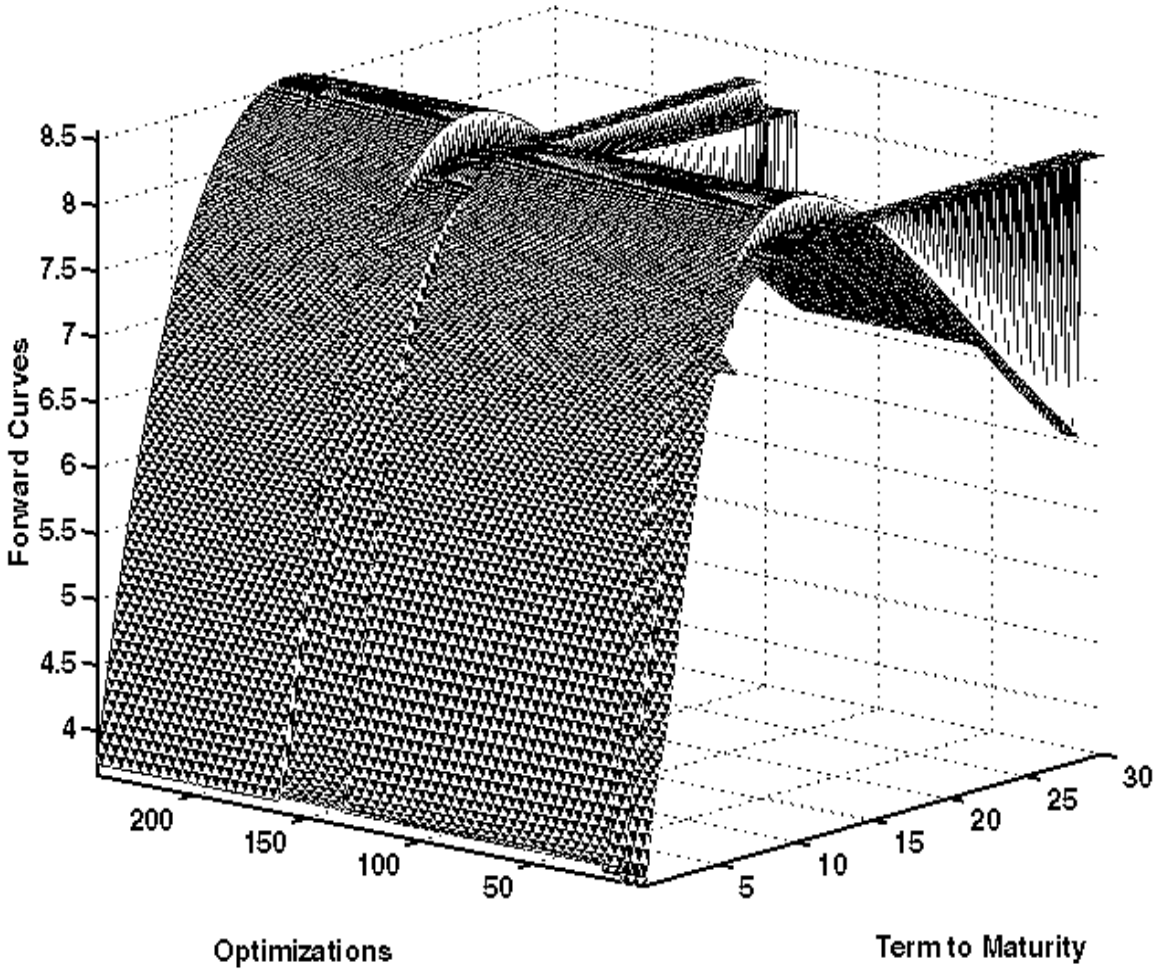
model.[16] For the results of the Svensson model, presented in Figure 4, only 11 of the 256 sets of starting parameters are close to the best curve.

Figure 3. Estimation of Nelson-Siegel forward curves for 17 January 1994 (81 different sets of starting parameters)



16. The definition of closeness to the best forward curve is based on estimated value for the objective function used in the estimation. An estimated curve is close to the best one when its estimated objective function value is at less than 0.1 per cent of the value of the highest objective function calculated. For further details on the objective functions used, see Technical Appendix, Section D, “Mechanics of the estimation,” on page 46.

Figure 4. Estimation of Svensson forward curves for 17 January 1994 (256 different sets of starting parameters)



In Section 4, the estimation issues are addressed directly by comparing the performance of the Nelson-Siegel, the Svensson, and the Super-Bell yield curve models in terms of the goodness of fit of the estimated curves to the Canadian data, the time required to obtain them, and their robustness to different strategies of optimization.

3. DATA

Prior to discussing the details of the various term structure models examined in this paper, it is necessary to describe Government of Canada bond and treasury bill data. The following sections briefly describe these instruments and the issues they present for the modelling of the term structure of interest rates.

3.1 Description of the Canadian data

The two fundamental types of Canadian-dollar-denominated marketable securities issued by the Government of Canada are treasury bills and Canada bonds.[17] As of 31 August 1998, the Government of Canada had Can$89.5 billion of treasury bills and approximately Can$296 billion of Canada bonds outstanding. Together, these two instruments account for more than 85 per cent of the market debt issued by the Canadian government.[18]

Treasury bills, which do not pay periodic interest but rather are issued at a discount and mature at their par value, are currently issued at 3-, 6-, and 12-month maturities. Issuance currently occurs through a biweekly "competitive yield" auction of all three maturities. The 6-month and 1-year issues are each reopened once on an alternating 4-week cycle and ultimately become fungible with the 3-month bill as they tend towards maturity. At any given time, therefore, there are approximately 29 treasury bill maturities outstanding.[19] Due to limitations in data availability, however, there is access only to 5 separate treasury bill yields on a consistent basis: the 1-month, 2-month, 3-month, 6-month, and 1-year maturities.

Government of Canada bonds pay a fixed semi-annual interest rate and have a fixed maturity date. Issuance involves maturities across the yield curve with original terms to maturity at issuance of 2, 5, 10, and 30 years.[20] Each issue is reopened several times to improve liquidity and achieve "benchmark status."[21] Canada bonds are currently issued on a quarterly "competitive yield" auction rotation with each maturity typically auctioned once per quarter.[22] In the interests of promoting liquidity, Canada has set targets for the total amount of issuance to achieve "benchmark status"; currently, these targets are Can$7 billion to Can$10 billion for each maturity. The targets imply that issues are reopened over several quarters in order to attain the desired liquidity.

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17. See Branion (1995) for a review of the Government of Canada bond market, and Fettig (1994) for a review of the treasury bill market.

18. The remaining market debt consists of Canada Saving Bonds, Real Return Bonds, and foreign currency denominated debt.

19. Effective 18 September 1997, the issuance cycle was changed from a weekly to a biweekly auction schedule. Previously, there were always at least 39 treasury bill maturities outstanding at any given time. The changes in the treasury bill auction schedule were designed to increase the amount of supply for each maturity by reducing the number of maturity dates that exist for treasury bills.

20. Canada eliminated 3-year bond issues in early 1997; the final 3-year issue was 15 January 1997.

21. A "benchmark" bond is analogous to an "on-the-run" U.S. Treasury security in that it is the most actively traded security for a given maturity.

22. It is important to note that Government of Canada bond yields are quoted on an Actual/Actual day count basis net of accrued interest. The accrued interest, however, is by market convention calculated on an Actual/365 day count basis. See Barker (1996) and Kiff (1996).

At any given time, therefore, there are at least four benchmark bonds outstanding with terms to maturity of approximately 2, 5, 10, and 30 years.[23] These bonds are the most actively traded in the Canadian marketplace. They are also often subject to underpricing in comparison with other Canada bonds because of a stronger demand. It could then be argued that these bonds should be excluded from the sample to avoid any downward bias in the estimation of the Canadian yield curve. Nevertheless, given that the Bank's main interest in estimating yield curves is to provide insights into the evolution of market expectations, it is considered essential that the information contained in these bonds be incorporated into the yield curve estimation. Therefore, all the benchmark bonds are forced to appear in all data sets.

Historically, the Government of Canada has also issued bonds with additional features on top of the "plain vanilla" structure just described. Canada has in the past issued bonds with callable and extendible features and a small number of these bonds remain outstanding. In addition, "purchase fund" bonds, which require periodic partial redemptions prior to maturity, were also issued in the 1970s. Finally, Real Return Bonds (RRB), which pay a coupon adjusted for changes in the Canadian consumer price index, were introduced in December 1991. There are two RRB maturities outstanding for a total of approximately Can$10 billion.[24] These bonds with unique features—purchase fund bonds and RRB—are flagged in the data base and subsequently excluded from the data set. Real Return Bonds are also excluded as their yields, which are quoted on a real rather than a nominal basis, are not directly comparable with nominal yields.[25]

3.2 Why are the data important?

The only bonds selected from the universe of Government of Canada bond and treasury bill data are those that are indicative of current market yields. This is because, regardless of the type of model selected, the results of a given yield curve model depend importantly on the data used to generate it. The examination of different filterings is therefore essential to provide confidence in the choice of the model and to ensure its efficient application. As a result, a system of filters is used to omit bonds that create distortions in the estimation of the yield curve. Analysis is centred on two specific aspects of data filtering that are considered strategic: the severity of the filtering (or its "tightness"), and the treatment of securities at the short-end of the term structure. The severity of filtering includes filters dealing with the maximum divergence from par value and the minimum amount outstanding required for the inclusion of a bond. The short-end of the term structure involves questions surrounding the inclusion or exclusion of treasury bills and bonds

23. As previously discussed, new issues may require two or more reopenings to attain "benchmark status." As a result, the decision as to whether or not a bond is a benchmark is occasionally a matter of judgment. This could lead to situations where more than one benchmark may exist for a given maturity.

24. See Côté, Jacob, Nelmes, and Whittingham (1996) for a discussion of the Canadian Real Return Bond.

25. There are approximately nine bonds with special features in the government's portfolio.

with short term to maturities. The two main filtering categories are considered in the following discussion.

3.3 Approaches to the filtering of data

3.3.1 Severity of data filtering: Divergence from par and amount outstanding

At present, there are 81 Canada bonds outstanding. This translates into an average issue size of roughly Can$3.5 billion. In reality, however, the amount outstanding of these bonds varies widely from Can$100 million to Can$200 million to just over Can$10 billion. Outstanding bonds of relatively small size relate to the previous practice of opening a new maturity for a given bond when the secondary market yield levels for the bond differed from the bond's coupon by more than 50 basis points. This is no longer the practice, given the benchmark program. At present, the current maturity is continued until the benchmark target sizes are attained irrespective of whether the reopening occurs at a premium or a discount. As bonds should have the requisite degree of liquidity to be considered, bonds with less than a certain amount outstanding should be excluded from the data set.[26] A relatively small value is assigned to the "minimum amount outstanding" filter in order to keep as many bonds as possible in the sample. One could argue, however, that only bonds with greater liquidity should be kept in the sample. This is an issue that will be investigated in the analysis of the data problem.[27]

The term "divergence from par value" is used to describe the possible tax consequences of bonds that trade at large premiums or discounts to their par value. Under Canadian tax legislation, interest on bonds is 100 per cent taxable in the year received, whereas the accretion of the bond's discount to its par value is treated as a capital gain and is only 75 per cent taxable and payable at maturity or disposition (whichever occurs first). As a result, the purchase of a bond at a large discount is more attractive, given these opportunities for both tax reduction and tax deferral. The willingness of investors to pay more for this bond, given this feature, can lead to price distortions. To avoid potential price distortions when large deviations from par exist, those bonds that trade more than a specified number of basis points at a premium or a discount from their coupon rate should be excluded.[28] The number of basis points selected should reflect a threshold at which the tax effect of a discount or premium is believed to have an economic impact.[29] The tax impact is

26. For example, if the specified minimum amount outstanding is Can$500 million, no bonds would be excluded on 15 June 1989 and eight bonds on 15 July 1998.

27. Of note, the amount outstanding of each individual issue could not be considered before January 1993 because of data constraints.

28. If this filter were set at 500 basis points, 8 bonds would be excluded on 15 June 1989 and 26 bonds on 15 July 1998.

29. See Litzenberger and Rolfo (1984).

somewhat mitigated in the Canadian market, however, as the majority of financial institutions mark their bond portfolios to market on a frequent basis. In this case, changes in market valuation become fully taxable immediately, thereby reducing these tax advantages somewhat. Moreover, some financial institutions are not concerned by these tax advantages.[30]

The divergence from par value and the amount outstanding filters are intimately related because bonds that trade at large discounts or premiums were typically issued with small amounts outstanding during transition periods in interest rate levels. Consequently, any evaluation or testing of this filter must be considered jointly with the minimum amount outstanding tolerated. These two filtering issues combined can then be identified as the severity or tightness of the filtering constraints. A looser set of divergence from par value and amount outstanding filtering criteria should provide robustness in estimation but can introduce unrepresentative data to the sample. Conversely, more stringent filtering criteria provide a higher quality of data but can lead to poor results given its sparsity. Specifically, tighter filtering reduces the number of observations and can make estimation difficult given the dispersion of the data. To cope with this data problem, the empirical analysis will include an evaluation of the sensitivity of the models' results to the degree of tightness chosen for these two filters.

3.3.2 The short-end: Treasury bills and short-term bonds

Choosing the appropriate data for modelling the short-end of the curve is difficult. Canada bonds with short terms to maturity (i.e., roughly less than two years) often trade at yields that differ substantially from treasury bills with comparable maturities.[31] This is largely due to the significant stripping of many of these bonds, which were initially issued as 10- or 20-year bonds with relatively high coupons leading to substantial liquidity differences between short-term bonds and treasury bills.[32] From a market perspective, these bond observations are somewhat problematic due to their heterogeneity in terms of coupons (with the associated coupon effects) and liquidity levels.

As a result of these liquidity concerns, one may argue for the inclusion of treasury bills in the estimation of the yield curve to ensure the use of market rates for which there is a relatively

30. For example, earnings of pension funds on behalf of their beneficiaries are taxable only at withdrawal from the pension accounts. Therefore, most investment managers of pension funds are indifferent to any tax advantage.

31. See Kamara (1990) for a discussion of differences in liquidity between U.S. Treasury bills and U.S. Treasury bonds with the same term to maturity.

32. In 1993, reconstitution of Government of Canada strip bonds was made possible in combination with the introduction of coupon payment fungibility. At that point in time, a number of long-dated high-coupon bonds were trading at substantial discounts to their theoretical value. The change in stripping practices played a substantial role in permitting the market to arbitrage these differences. See Bolder and Boisvert (1998) for more information on the Government of Canada strip market.

high degree of confidence. Treasury bills are more uniform, more liquid, and do not have a coupon effect given their zero-coupon nature. The question arises as to whether or not to use short-term bonds and/or treasury bills in the data sample. Using only treasury bills would avoid the estimation problems related to the overwhelming heterogeneity of coupon bonds at the short-end of the maturity spectrum and anchor the short-end of the yield curves by the only zero-coupon rates that are observed in the Canadian market.

Recent changes in the treasury bill market have nonetheless complicated data concerns at the short-end of the curve. Declining fiscal financial requirements have led to sizable reductions in the amount of treasury bills outstanding. In particular, the stock of treasury bills has fallen from $152 billion as at 30 September 1996 to $89.5 billion as at 31 August 1998. This reduction in stock with no corresponding reduction in demand has put downward pressure on treasury bill yields.[33] This raises concerns about the use of treasury bills in the data sample. This data problem will also be addressed in the empirical analysis, by an estimation of the sensitivity of the models' results to the type of data used to model the short-end of the maturity spectrum.

4. EMPIRICAL RESULTS

To perform an empirical analysis of the behaviour of the different yield curve models and their sensitivity to data filtering conditions, a sample of 30 dates has been chosen, spanning the last 10 years. The dates were selected to include 10 observations from an upward-sloping, a flat, and an inverted term structure environment. This helps to give an understanding of how the model performs under different yield curve slopes. The following table (Table 1) outlines the various dates selected. It is worth noting that these dates could not be randomly selected as there are only a few instances in the last 10 years of flat or inverted Canadian term structure environments. As a result, the flat and inverted term structure examples are clustered around certain periods.

33. See Boisvert and Harvey (1998) for a review of recent developments in the Government of Canada on the treasury bill market.

Table 1. Dates selected for estimation from different term structure environments

Positively sloped term structure	Flat term structure	Inverted term structure
15 February 1993	15 August 1988	15 January 1990
15 July 1993	18 August 1988	15 May 1990
17 January 1994	23 August 1988	15 August 1990
16 May 1994	29 August 1988	13 December 1990
15 August 1994	15 February 1991	14 April 1989
15 February 1995	25 February 1991	15 June 1989
17 July 1995	4 March 1991	15 August 1989
15 February 1996	11 March 1991	16 October 1989
15 August 1996	15 June 1998	15 December 1989
16 December 1996	15 July 1998	15 March 1990

As discussed in Section 3.3, there is a wide range of possible data filtering combinations that could be analyzed and their interaction is complex. As a result, examination has been limited to a few dimensions. To do so, first a "benchmark" filtering case is defined, based on a set of preliminary choices for each type of filtering criteria. The benchmark case is summarized as follows:

Table 2. Filter settings for the benchmark case

Type of data filter	Filter setting
Minimum amount outstanding	Can$500 million
Divergence from par: | Coupon - YTM |	500 basis points
Inclusion of treasury bills	Yes
Inclusion of bonds with less than 2 years TTM	No

This benchmark data filtering case is held constant for a variety of different estimation approaches (detailed in Section 4.1) and deals explicitly with the estimation problem. After this analysis is complete, the best optimization approach is selected and used to consider three alternative data filtering scenarios. Each of these alternatives is contrasted in Section 4.2 with the benchmark case to examine the models' sensitivity to the two main aspects that were discussed in the previous section. Thus the estimation problem is considered while holding constant the data issue, and the data problem is subsequently examined holding the estimation problem constant.

4.1 The "estimation problem"

As illustrated in Section 2.2, the Nelson-Siegel and Svensson models are sensitive to the estimation procedure chosen and particularly to the starting values used for the parameters. Moreover, the time required to increase the robustness of an estimated curve, or the confidence of

having a global minimum, increases exponentially with the number of different starting values chosen for each parameter. To address this problem of estimation, a number of procedures are examined to find a reasonable solution to this trade-off between time, robustness, and accuracy. Specifically, the strategy for dealing with the estimation problem was to consider a number of different approaches to the problem for the 30 different dates chosen and to examine the results using the benchmark selection of the data. The Nelson-Siegel and Svensson curves are not determined in a statistical estimation but rather in a pure optimization framework. Therefore, an objective function must be specified and subsequently minimized (or maximized), using a numerical optimization procedure. Consequently, the approaches differ in terms of the formulation of the objective function and the details of the optimization algorithm.

Two alternative specifications of the objective function are examined. Both approaches seek to use the information in the bid-offer spread. One uses a *log-likelihood* specification while the other minimizes a special case of the weighted *sum of squared price errors*. The log-likelihood formulation replaces the standard deviation in the log-likelihood function with the bid-offer spread from each individual bond. The sum of squared price error measure puts a reduced weight on errors occurring inside the bid-offer spread but includes a penalty for those observations occurring outside the bid-offer spread. These two formulations are outlined in greater detail in the Technical Appendix.

Each optimization algorithm can be conceptually separated into two parts: the global and local search components. The global search is defined as the algorithm used to find the appropriate region over the domain of the objective function. The distinction is necessary due to the widely varying parameter estimates received for different set of starting values. The intent is to broadly determine a wide range of starting values over the domain of the function and then run the local search algorithm at each of these points. The local search algorithm finds the solution from each set of starting values using either Sequential Quadratic Programming (a gradient-based method) and/or the Nelder and Meade Simplex Method (a direct search, function-evaluation-based method). Two basic global search algorithms are used:

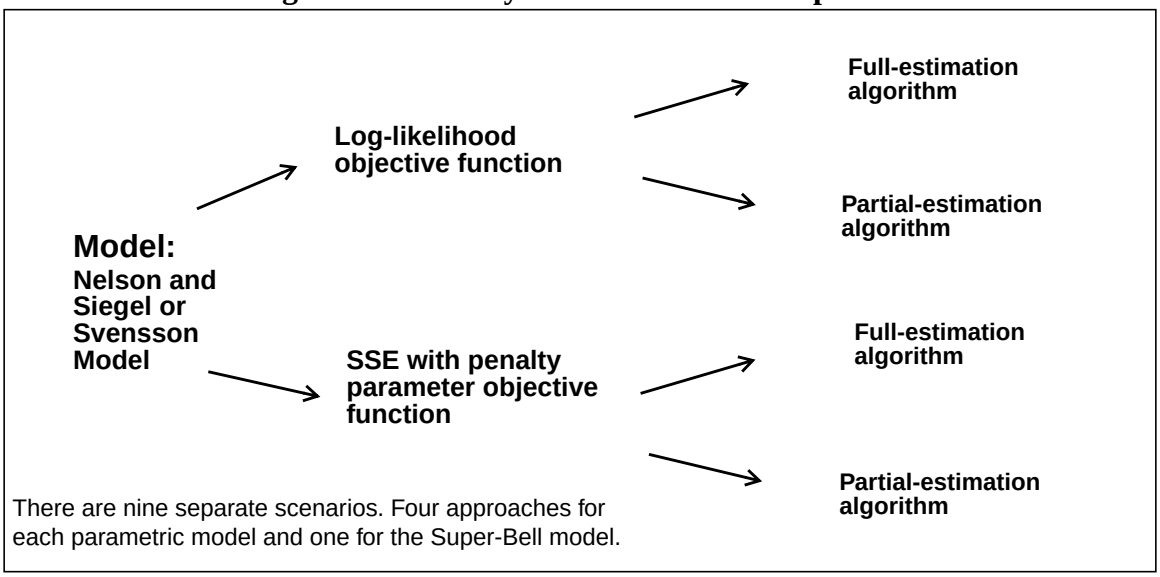
- ***Full estimation (or "coarse" grid search):*** This approach uses a number of different sets of starting values and runs a local search for each set and then selects the best solution. In both the Nelson-Siegel and Svensson models, the β_0 and β_1 parameters were not varied but rather set to the long-run term to maturity and the difference between the longest and shortest yield to maturity. In the Nelson-Siegel model, therefore, 9 combinations of the remaining 2 parameters (β_2 and τ_1) are used in the grid for a total of 81 sets of starting parameters. In the Svensson model, there are 4 combinations of 4 parameters (β_2, β_3, τ_1, τ_2) for a total of 256 starting values. In the full-estimation algorithm, the Sequential

Quadratic Programming (SQP) algorithm is used; this is replaced by the Simplex method when the SQP algorithm fails to converge.[34]

- ***Partial estimation:*** The second approach uses partial estimation of the parameters. Specifically, this global search algorithm divides the parameters into two groups, the βs (or linear parameters) and the τs (or the non-linear parameters). The algorithm works in a number of steps where one group of parameters is fixed while the other is estimated.[35] The full details of this algorithm are presented in the Technical Appendix, Section E.2, "Partial-estimation algorithm," on page 52.

In total, four separate approaches to parameter estimation are examined for each of the two parametric models: two separate formulations of the objective function and two separate global search algorithms. The estimation of the parameters for the Super-Bell model is a simple matter of OLS regression. This means that, while there are only three models from which to select, there is a total of nine sets of results (this is depicted graphically in Figure 5).

Figure 5. The analysis of the "estimation problem"



The use of a numerical optimization procedure neither provides standard error measures for parameter values nor permits formal hypothesis testing. Instead, therefore, the approach involves a comparison among a variety of summary statistics. Three main categories of criteria have been selected: goodness of fit, speed of estimation, and robustness of the solution. A number

34. This idea comes from Ricart and Sicsic (1995) although they actually impose these as constraints. In this paper, they are used as starting points. See Technical Appendix, Section E.1, "Full-estimation algorithm," on page 50 for more detail.

35. In the partial-estimation algorithm, the SQP algorithm is used exclusively because there were no convergence problems when estimating a smaller subset of parameters.

of different statistics were selected to assess the performance of each of the approaches under each of these categories. The following sections discuss and present each group of criteria in turn.

4.1.1 Robustness of solution

Robustness of solution can be defined as how certain one is that the final solution is actually the global minimum or maximum. This measurement criterion is examined first because it provides an understanding of the differences and similarities between the optimization strategies. Two measures of robustness are considered.

The first measure, in Table 3, compares the best objective function values for each of the alternative optimization approaches. Only objective function values based on the same model with the same estimation algorithm are directly comparable (i.e., one compares the figures in Table 3 vertically rather than horizontally). Consequently, Table 3 compares the full- and partial-estimation algorithms for each formulation of the objective function. A number of observations follow:

- In all cases, save one, the Nelson-Siegel partial- and full-estimation algorithms lead to the same results. The one exception is the full-estimation algorithm, which provides a superior value for the sum of squared errors objective function on 18 August 1988.

- The Svensson model full-estimation algorithm provides in all cases a superior or identical result to the partial-estimation algorithm. The full-estimation algorithm outperforms the partial on eight occasions for the log-likelihood objective function and on seven occasions for the sum of squared errors objective function.

- The magnitude of a superior objective function value is also important. In aggregate, the differences in objective function are quite small and it will be important to look to other statistics to see the practical differences (particularly the goodness of fit) in the results of these different solutions.

Table 3. Best objective function value

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	100,707.6	100,707.6	86,799.7	87,755.1	51,813.5	51,813.5	23,997.7	25,908.4	n/a
Flat	61,707.3	61,707.3	54,616.2	54,837.2	97,587.4	97,743.8	83,405.1	83,643.4	n/a
Inverted	24,006.0	24,006.0	21,016.5	21,432.2	40,660.5	40,660.5	34,681.5	35,182.6	n/a
Total	62,140.3	62,140.3	54,144.1	54,674.8	63,353.8	63,406.0	47,361.5	48,244.8	n/a

The second statistic to consider is the number of solutions in the global search algorithm that converge to within a very close tolerance (0.01 per cent) of the best solution. Table 4 outlines the aggregate results.

- This is a rather imperfect measure for comparison among global search algorithms because the partial estimation fixes sets of parameters, which necessarily constrain it from the optimal solution. It is nonetheless useful for comparison between different specifications of the objective function, between different models, and between different term structure environments.

- A not-surprising result is that the simpler Nelson-Siegel model has a much higher rate of similar solutions (approximately 60 per cent for the full estimation versus approximately 30 per cent for the Svensson model).

- It appears more difficult to estimate an upward-sloping term structure than one that is flat or inverted. For the full-estimation algorithm, the flat and inverted term structures have roughly twice as many similar solutions as in the upward-sloping yield curve environment.

- The data do not suggest a substantial difference between the two alternative formulations of the objective function.

Table 4. Percentage of solutions in global search within 0.01 per cent of best solution

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	58.8%	10.0%	17.1%	6.1%	58.6%	5.7%	16.8%	1.8%	n/a
Flat	53.1%	17.3%	35.8%	5.4%	52.5%	1.3%	35.1%	8.9%	n/a
Inverted	76.8%	23.0%	35.1%	9.8%	76.3%	34.0%	32.9%	2.4%	n/a
Total	62.8%	16.8%	29.3%	7.1%	62.5%	17.5%	28.2%	4.4%	n/a

4.1.2 Goodness of fit

This is arguably the most important of the three criteria because these measures indicate how well the model and its associated estimation procedure describe the underlying data. It should be stated at the beginning that, in those instances in the previous section, particularly for the Nelson-Siegel model where the solutions were identical, the results will be identical. This section will therefore focus on the differences between models, objective functions, and optimization strategy where appropriate. Five different measures have been selected to determine the "fit" of the various strategies. The measures focus on yield errors. This is important because, although price is used in the estimation of the models, it is appropriately weighted to ensure a good fit to the bond yield to maturity.

Table 5 displays the first measure of goodness of fit, the yield root mean square error *($RMSE_{yield}$)*.[36] In a general sense, this measure can be interpreted as the standard deviation of the yield errors.

- In aggregate, the Svensson model appears to perform about one basis point better than the Nelson-Siegel model.

- The data also suggest that all the models do a superior job of fitting an upward-sloping term structure relative to their flat or inverted counterparts. Caution is suggested in this assessment, given the relatively skewed nature of the sample selection. There may be reason to suspect that the periods from which the inverted and flat term structure dates were selected are different from the broader range of dates selected for the upward-sloping term structure.

36. The root mean square error is defined as $RMSE_{yield} = \sum_{i=1}^{N} \sqrt{\frac{(e_{i,yield} - \bar{e}_{yield})^2}{n}}$.

- Despite the differences in the solutions between the full- and partial-estimation algorithm of the Svensson model, the results are quite similar. Indeed, the full-estimation algorithm provides an improvement of only one-tenth of a basis point over the partial estimations. This would suggest that the practical differences among the solutions are not large.

- Note that the Svensson and Nelson-Siegel models provide a marked improvement relative to the Super-Bell model in the upward-sloping and inverted term structure environments but are roughly similar when the term structure is flat.

Table 5. Root mean square yield error (in basis points)

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	8.6	8.6	6.5	6.5	8.8	8.6	6.5	6.5	11.3
Flat	25.5	25.5	24.4	25.0	25.5	25.5	24.4	24.9	25.7
Inverted	18.1	18.1	18.5	18.5	18.1	18.1	18.5	18.5	26.2
Total	17.4	17.4	16.5	16.7	17.4	17.4	16.5	16.6	21.1

The $RMSE_{yield}$ measure is essentially a standard-deviation-based measure that uses the squared deviations from the mean as its numerator. As a consequence, it is rather sensitive to outliers. For this reason, an alternative measure of yield errors is also examined: the average absolute value of yield errors *(*$AABSE_{yield}$*)*.[37] This measure is less sensitive to extreme points. The results of this measure are illustrated in Table 6.

- Given the reduced sensitivity to large values, it is hardly surprising to note that the errors are in general somewhat smaller (roughly about five basis points for the total sample). The fact that the differences between $RMSE_{yield}$ and the $AABSE_{yield}$ are larger for the flat and inverted dates than for the normal dates suggests that there are more outliers occurring on these dates.

- In general, however, the same relationships that appear in the $RMSE_{yield}$ are evident in these results. That is, the Svensson model slightly outperforms the Nelson-Siegel model and the upward-sloping term structure is a better fit than the flat and inverted yield curve environments.

37. The average absolute value of yield errors is defined as: $AABSE_{yield} = \sum_{i=1}^{N} \frac{|e_{i,yield} - \bar{e}_{yield}|}{n}$.

- It is interesting to note that the Svensson model full-estimation algorithm fits marginally less well relative to the partial-estimation algorithm (on a margin of one-tenth of one basis point). This is the opposite of the results found using $RMSE_{yield}$ measure.

- Again, both the Svensson and Nelson-Siegel models provide a substantially improved fit over the Super-Bell model (on aggregate by three to four basis points) in upward-sloping and inverted environments. All models appear to perform similarly in a flat term structure setting.

Table 6. Average absolute value of yield errors (in basis points)

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	6.6	6.6	5.1	5.0	6.6	6.6	5.1	5.0	9.5
Flat	17.4	17.4	16.4	16.8	17.4	17.5	16.4	16.7	17.0
Inverted	13.4	13.5	13.1	13.1	13.4	13.4	13.1	13.1	18.4
Total	12.5	12.5	11.5	11.6	12.5	12.5	11.5	11.6	15.0

The next measure of goodness of fit is termed the hit ratio. This statistic describes the number of bonds with an estimated price inside the bid-offer spread as a percentage of the total number of bonds estimated. The intent of this measure is to get a sense of the number of bonds that were essentially perfectly priced. This is particularly interesting when considering the formulation of the objective function measures, which explicitly use the bid-offer spread. Table 7 illustrates the results.

- The hit ratio is roughly two times higher for the upward-sloping relative to the flat and inverted term structures. That is to say, approximately twice as many estimated bond yields fall between the bid and offer spread for the upward-sloping term structure observations.

- The Nelson-Siegel model appears to perform better than the Svensson model for the flat and inverted term structures and worse for an upward-sloping yield curve. In aggregate, they even out and show little difference.

- Once again, in all cases, the Nelson-Siegel and Svensson models outperform the Super-Bell model on this measure except for the flat term structure dates.

Table 7. Hit ratio (estimated yield inside bid-offer spread/total bonds)

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	9.7%	9.7%	11.5%	11.5%	9.7%	9.7%	11.4%	11.5%	6.9%
Flat	4.3%	4.3%	3.4%	3.2%	4.3%	4.2%	3.7%	3.5%	3.8%
Inverted	5.0%	5.0%	3.8%	4.1%	5.0%	5.0%	4.0%	4.2%	2.1%
Total	6.3%	6.3%	6.3%	6.3%	6.3%	6.3%	6.3%	6.4%	4.3%

The following two tables, Tables 8 and 9, provide a sense of whether or not the estimates are biased in one direction or another. These two measures, which are essentially rough measures of dispersion, describe the percentage of estimated yields exceeding the bid yield and the percentage of estimated yields below the offer yield.

- The Super-Bell model tends to overestimate yields to maturity or, alternatively, to underestimate bond prices.

- For upward-sloping yield curves, the Nelson-Siegel model tends to underestimate the actual yields to maturity, while the Svensson model does not seems to be biased in a direction.

Table 8. Percentage of bonds with estimated yields exceeding the bid

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	47.0%	47.0%	50.0%	48.4%	47.0%	47.0%	50.2%	48.7%	55.8%
Flat	47.5%	47.5%	46.5%	46.9%	47.5%	47.9%	46.6%	47.0%	52.5%
Inverted	56.3%	56.3%	53.2%	53.5%	56.3%	56.3%	53.2%	53.7%	60.8%
Total	50.3%	50.3%	50.0%	49.6%	50.3%	50.4%	50.0%	49.8%	56.4%

- For flat curves, both the parametric models tend to underestimate the actual yields to maturity, while they seem to underestimate yields to maturity for inverted curves, although by a lesser amount than the Super-Bell model.

- In general, the Svensson model appears to perform slightly better than the Nelson-Siegel model with less tendency to be biased in one direction.

Table 9. Percentage of bonds with estimated yields below the offer

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	53.0%	53.0%	50.0%	51.6%	53.0%	53.0%	49.8%	51.3%	44.2%
Flat	52.5%	52.5%	53.5%	53.1%	52.5%	52.1%	53.4%	53.0%	47.5%
Inverted	43.7%	43.7%	46.8%	46.5%	43.7%	43.7%	46.8%	46.3%	39.2%
Total	49.7%	49.7%	50.0%	50.4%	49.7%	49.6%	50.0%	50.2%	43.6%

4.1.3 Speed of estimation

The final criterion, the speed of optimization, is one of large practical importance. Given the finite amount of time and computing resources available to solve this problem, a fair amount of importance will be placed on this criterion.[38]

The first measure examined is the average amount of time required for each individual local search within the larger global search algorithm. The full- and partial-estimation algorithms cannot be explicitly compared, given that full estimation has a much harder task (for each individual iteration) relative to the partial-estimation approach. It nevertheless provides some interesting information regarding the differences between the two techniques. Table 10 details this measure.

- The full-estimation algorithm, for both models and objective function values, appears to take on average 10 times longer per iteration than the partial-estimation approach.

- The Svensson model requires approximately four times as much time compared with the Nelson-Siegel model for both objective function formulations and global estimation algorithms.

- It does not appear that there are substantial differences in the average amount of time required per iteration for the different term structure environments.

- The log-likelihood objective function is slightly faster (on the order of about one second) for both the Nelson-Siegel and Svensson models.

38. All the estimations were performed using a Sun Microsystems Ultra 10 workstation and the mathematical software, Matlab.

Table 10. Average time per local search algorithm (in seconds)

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	11.3	1.7	47.5	4.1	11.5	1.6	48.5	4.4	0.5
Flat	10.7	2.0	49.0	4.6	11.1	2.2	50.3	6.2	0.5
Inverted	13.9	2.1	46.3	4.5	14.2	2.2	47.3	4.6	0.5
Total	12.0	1.9	47.6	4.4	12.3	2.0	48.7	5.1	0.5

Table 11 summarizes the final measure to be considered: the total amount of time per global search. It is worth noting that, while this statistic gives a good approximation of the amount of time required for the partial-estimation algorithm, it tends to underestimate the time required for the full-estimation algorithm. Specifically, it has not been possible to capture the time required by the SQP algorithm in those instances where the SQP algorithm did not converge and was replaced by the Simplex algorithm.

- In aggregate, the full estimations take roughly six times longer than the partial-estimation algorithms.

- As was the case with the previous measure, there do not appear to be substantial differences in the time required for estimation of different term structure environments.

- The log-likelihood and sum of squared error objective functions require approximately the same amount of time for the full-estimation algorithm. The partial-estimation algorithm, however, appears to be marginally faster (10 to 15 minutes) for the log-likelihood function.

Table 11. Total time for global search algorithm (in hours)

Dates	Nelson-Siegel model		Svensson model		Nelson-Siegel model		Svensson model		Super-Bell model OLS estimation
	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	Full-estimation algorithm	Partial-estimation algorithm	
	Log-likelihood objective function				Sum of squared errors objective function				
Normal	0.25	0.17	3.20	0.44	0.26	0.18	3.27	0.60	0.00013
Flat	0.24	0.37	3.31	0.54	0.25	0.30	3.40	0.88	0.00013
Inverted	0.31	0.47	3.13	0.66	0.32	0.43	3.20	0.71	0.00013
Total	0.27	0.33	3.21	0.55	0.28	0.30	3.29	0.73	0.00013

4.1.4 The "estimation" decision

As indicated, the final step in the "estimation problem" is to select the most promising model and optimization strategy in order to examine two specific aspects of filtering the data. The model ultimately selected was the Svensson model with a log-likelihood objective function using the partial-estimation algorithm. There were several reasons for this decision:

- Although the full-estimation procedure appears to provide slightly better solutions than the partial-estimation algorithm, the resulting goodness-of-fit measures were not different in a practical sense.

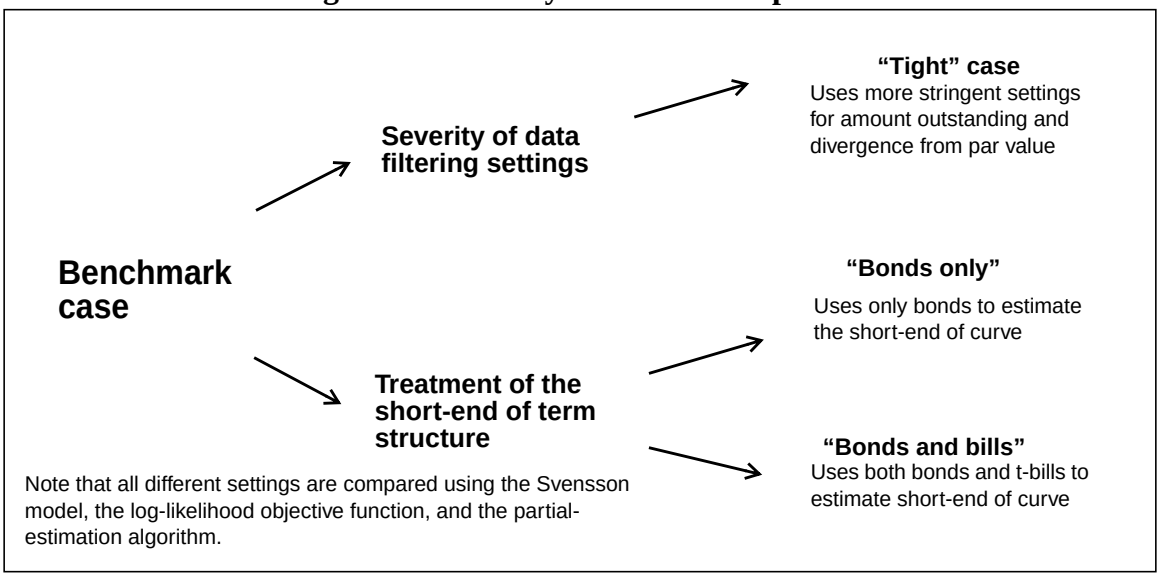
- The full-estimation algorithm is prohibitively time consuming. On average, by conservative measures, the full procedure required roughly six times longer than the partial-estimation procedure.

- There does not appear, in the statistics considered, to be much in the way of practical difference between the two objective function formulations. As a result, the decision is somewhat arbitrary. The log-likelihood specification was finally selected because it is slightly faster for the partial-estimation algorithm.

4.2 The "data problem"

This section includes a sensitivity analysis of two aspects of data filtering that are considered important in the Canadian data: the severity of the filtering criteria, and the treatment of the short-end of the term structure. Accordingly, the analysis performed in this section compares the results obtained with the benchmark filtering relative to three alternative filtering settings: a scenario with a more severe (or "tight") setting, one with only bonds included at the short-end, and one with both bonds and treasury bills at the short-end (see Figure 6). The settings for the benchmark data filtering are outlined in Table 2. The different filtering is compared using the best optimization approach from the previous section, that is, the Svensson yield curve with a log-likelihood objective function using the partial-estimation algorithm for the same 30 dates used in the initial analysis.

Figure 6. The analysis of the "data problem"



To ensure the comparability among the different filtering settings and models, the same summary statistics as those presented in the previous section are used: goodness of fit, robustness, and speed of estimation. In addition, these statistics are calculated using the same unfiltered observations as in the benchmark case, rather than the actual unfiltered observations of the various alternative filterings.

The average number of unfiltered observations for each shape of curve for the different filterings applied is listed in Table 12. The use of both bonds and bills at the short-end has the most observations while the "tight" case, not surprisingly, has fewer observations relative to the alternatives. The fact that there are more observations in the "bonds only" case suggests there are more bonds at the short-end of the term structure than the five treasury bill observations used in the benchmark case.

Table 12. Number of observations used in estimation (unfiltered observations)

Dates[a]	Benchmark case	"Bonds and bills" case	"Bonds only" case	"Tight" case
Normal	52.3	61.2	56.2	26.2
Flat	74.5	94.2	89.2	56.5
Inverted	83.3	101.7	97.2	65.5
Total	70.0	85.7	80.9	49.4

a. Of note, the average number of available observations for normal, flat, and inverted curves is 95.8, 119.4, and 125.8 respectively, for a total average of 113.7 observations.

4.2.1 Tightness of data filtering

The first filtering issue deals with the severity of the filtering constraints. Given that the benchmark case filtering settings are not particularly stringent, a more severe or "tight" set of filtering criteria is analyzed (see Table 13). Note that the minimum amount outstanding has been increased to $2.5 billion and the divergence from par value to 250 basis points. While these settings are clearly debatable, they do represent a substantial tightening from the benchmark case.

Table 13. Filter settings: "Tight" case

Type of data filter	"Tight" filtering	"Benchmark" filtering
Minimum amount outstanding	Can$2,500 million	Can$500 million
Divergence from par: | Coupon - YTM |	250 basis points	500 basis points
Inclusion of treasury bills	Yes	Yes
Inclusion of bonds with less than 2 years TTM	No	No

Table 14 illustrates the summary statistics for the goodness of fit. The following observations are made:

- The tight filtering tends to outperform the benchmark case filtering for the $RMSE_{yield}$ and $AABSE_{yield}$ measures in flat or inverted term structures although they are broadly similar in an upward-sloping environment.

- When considering the hit ratio, however, there does not appear to be a significant difference between the two filtering options.

- The benchmark case filtering performs marginally better for upward-sloping yield curves.

Table 14. Goodness of fit: "Tight" vs. benchmark

Dates	Yield root mean square error (in basis points)		Average absolute value of yield errors (in basis points)		Hit ratio (%)	
	Benchmark filtering	Tight filtering	Benchmark filtering	Tight filtering	Benchmark filtering	Tight filtering
Normal	6.5	6.9	5.0	5.4	11.5%	11.9%
Flat	25.0	19.6	16.8	13.7	3.2%	3.1%
Inverted	18.5	15.8	13.1	11.7	4.1%	5.4%
Total	16.7	14.1	11.6	10.2	6.3%	6.8%

By reviewing the statistics on the speed of estimation (presented in Table 15), the following conclusions can be made:

- The average time taken per local search algorithm is generally very close for the two alternatives.

- Similarly, the total time used for the global search algorithm is broadly comparable for both filterings. Nevertheless, the tight filtering takes slightly more time on flat curve estimations while the benchmark filtering is somewhat slower on inverted curve estimations.

Table 15. Speed of estimation: "Tight" vs. benchmark

Dates	Average time per local search algorithm (in seconds)		Total time for global search algorithm (in hours)	
	Benchmark filtering	Tight filtering	Benchmark filtering	Tight filtering
Normal	4.1	4.1	0.44	0.44
Flat	4.6	4.8	0.55	0.67
Inverted	4.5	4.5	0.66	0.60
Total	4.4	4.5	0.55	0.57

Finally, the results of the robustness statistics, as illustrated in Table 16, are as follows:

- On average, the tight filtering attains a superior objective function value relative to the benchmark filtering. This confirms the generally better fit observed in the goodness-of-fit statistics.

- The percentage of estimated objective functions that are within 0.1 per cent of the best value obtained is of similar magnitude in both filtering cases.

Table 16. Robustness: "Tight" vs. benchmark

Dates	Best objective function values		Percentage of solutions in global search within 0.01% of best solution	
	Benchmark filtering	Tight filtering	Benchmark filtering	Tight filtering
Normal	87,755.1	87,732.9	21.6%	20.2%
Flat	54,837.2	49,560.0	18.2%	17.9%
Inverted	21,432.2	20,352.3	31.3%	31.9%
Total	54,674.8	52,548.4	23.7%	23.3%

4.2.2 Data filtering at the short-end of the term structure

At the short-end of the curve, the data filtering question involves the inclusion of treasury bills relative to the use of bond observations. To examine robustness of the model to the data selected for this sector of the curve, three possible alternatives are considered: only treasury bills (the benchmark case), only bonds, and both bonds and treasury bills. The two new alternatives—termed “bonds only” and “bonds and bills”—have the following settings:

Table 17. Filter settings: “Bonds only” and “bonds and bills”

Type of data filter	“Bonds only” filtering	“Bonds and bills” filtering	“Benchmark” filtering
Minimum amount outstanding	Can$500 million	Can$500 million	Can$500 million
Divergence from par: | Coupon - YTM |	500 basis points	500 basis points	500 basis points
Inclusion of treasury bills	No	Yes	Yes
Inclusion of bonds with less than 2 years TTM	Yes	Yes	No

Comparing these two filtering alternatives to the benchmark case on the basis of goodness of fit (see Table 18), it is observed that:

- There is no clear winner among the two principal yield error measures. For upward-sloping yield curves, there are smaller errors when both “bonds and bills” are used at the short-end of the maturity spectrum. Benchmark filtering generally outperforms the two alternatives for flat term structures. Finally, in an inverted environment, the benchmark case is superior when considering the $RMSE_{yield}$ while the bond and bill case is the best when using $AABSE_{yield}$. The differences are nonetheless in most instances quite small.

- The hit ratio appears to favour the benchmark filtering for upward-sloping curves. The "bonds only" case is the clear winner in flat and inverted term structure environments.

Table 18. Goodness of fit: Short-end vs. benchmark

Dates	Yield root mean square error (in basis points)			Average absolute value of yield errors (in basis points)			Hit ratio (%)		
	Bench-mark	Bonds only	Bonds and bills	Bench-mark	Bonds only	Bonds and bills	Bench-mark	Bonds only	Bonds and bills
Normal	6.5	6.8	6.4	5.0	5.3	4.8	11.5%	14.4%	14.0%
Flat	25.0	28.5	27.3	16.8	15.8	15.2	3.2%	7.8%	5.9%
Inverted	18.5	19.8	17.2	13.1	13.9	13.0	4.1%	6.0%	5.0%
Total	16.7	18.4	17.0	11.6	11.7	11.0	6.3%	9.4%	8.3%

The speed-of-estimation measures are detailed in Table 19.

- The average time per local search algorithm is faster in all cases under the benchmark case.
- This translates into a much lower total time of estimation for the benchmark filtering case than for the other two alternatives. Notably, the estimations using only bonds at the short-end of the maturity spectrum take twice as much time relative to the benchmark filtering.

Table 19. Speed of estimation: Short-end vs. benchmark

	Average time per local search algorithm (in seconds)			Total time for global search algorithm (in hours)		
Dates	Benchmark	Bonds only	Bonds and bills	Benchmark	Bonds only	Bonds and bills
Normal	4.1	5.2	4.4	0.44	1.23	0.95
Flat	4.6	4.9	4.9	0.55	1.08	1.09
Inverted	4.5	5.0	5.2	0.66	1.17	0.96
Total	4.4	5.1	4.8	0.55	1.16	1.00

Table 20 details the robustness criteria.

- Surprisingly, the best objective function values are the lowest for the "bonds and bills" filtering and the highest for the benchmark filtering.
- For all the types of shapes of yield curve, the benchmark filtering is the most robust of the three filterings in terms of the percentage of estimated values of the objective function within 0.1 per cent of the best value.

Table 20. Robustness: Short-end vs. benchmark

	Best objective function value			Percentage of solutions in global search within 0.01% of best solution		
Dates	Benchmark	Bonds only	Bonds and bills	Benchmark	Bonds only	Bonds and bills
Normal	87,755.1	75,149.9	36,369.9	21.6%	13.7%	10.6%
Flat	54,837.2	45,207.7	61,362.5	18.2%	12.7%	8.6%
Inverted	21,432.2	16,857.1	24,700.8	31.3%	22.9%	21.1%
Total	54,674.8	45,738.2	40,811.1	23.7%	16.4%	13.4%

4.2.3 The "data" decision

At first glance, the results tend to lead towards the use of tighter data filtering. The tighter filtering produces a better fit of the data for flat and inverted curves, at no cost in terms of speed and robustness of the estimations. However, the choice is less obvious if upward-sloping, or "normal," yield curves are considered, where the benchmark case still slightly outperforms the tight filtering in terms of goodness of fit. Moreover, the hit ratio statistics suggest that the benchmark filtering generates more estimated YTMs in the actual bid-offer spread. Since the process of choosing a threshold value for any filtering criteria is a somewhat arbitrary process, the benchmark filtering is still considered more reliable because it provides similar results while using more information from the government securities market.

The various criteria suggest that the "bonds only" or "bonds and bills" cases do not provide any clear improvement relative to the benchmark case in terms of goodness of fit. The analysis confirms the difficulty of using only the information embedded in short-term bonds to estimate the Svensson model. The slower global algorithm convergence times suggest that the use of either "bonds only" or "bonds and bills" at the short-end is more difficult. This is supported by the smaller number of solutions in the global search close to the best solution for both "bonds only" and "bonds and bills" relative to the benchmark case. It was therefore decided that the benchmark case seems a better approach. This may be because the sole use of treasury bills at the short-end, as in the benchmark case, helps anchor the model because these securities are more liquid and homogeneous than the existing bonds in this maturity area.

5. CONCLUDING REMARKS

The objectives of this paper were to introduce a new class of parametric term structure models to the Bank of Canada and to prepare the framework for the generation of a historical data base of Government of Canada yield curves. To tackle these issues, the problem was divided into two separate components: the estimation and the data aspects. In the analysis of the estimation problem, the data filtering criteria were held constant and three separate models—two alternative specifications of the objective function and two global search algorithms—were examined. Each of the nine alternatives was measured in terms of goodness of fit, speed of estimation, and robustness of the results. The best alternative was determined to be the Svensson model using a log-likelihood objective function and the partial-estimation algorithm. This estimation approach was then used to consider the data problem. To achieve this, three alternative filtering settings were considered: a more severe or "tight" setting and an examination of the use of bonds and/or treasury bills to model the short-end of the term structure. Once again, the goodness of fit, robustness, and speed of estimation were used to compare these different filtering possibilities. In the final analysis, it was decided that the benchmark filtering setting offered the best approach to the selection of data for the estimation of the term structure.

From this work emerges a framework for the development of a historical data base of estimated term structures and an improved understanding of this class of parametric models. In particular, there are a number of concerns respecting these models that have been resolved by this analysis. For example, it is believed that the log-likelihood specification of the objective function is an efficient approach to solving this problem. In addition, the benchmark data filtering case performs well relative to other possible filtering scenarios. Indeed, the parametric class of models appears to be less sensitive to the data filtering than initially believed. Some questions, however, remain. The first observation is that the estimation algorithms could be improved. There is a concern that the domain of the objective function is not adequately considered when determining the optimal set of starting parameters. A possible avenue of future research to deal more appropriately with the high dimensionality of the problem could involve the use of genetic algorithms. Finally, although the Svensson model was chosen, there are other functional forms that may be more stable or may better describe the underlying data. These two remaining questions, therefore, suggest that there are certainly more research questions to be addressed in this area.

TECHNICAL APPENDIXES

This appendix is divided into four sections:

- Section A: Basic “yield curve” building blocks
- Section B: Extracting zero-coupon rates from the par yield curve
- Section C: Extracting “implied” forward rates from zero-coupon rates
- Section D: Mechanics of the estimation
- Section E: Optimization algorithms

A. Basic “yield curve” building blocks[1]

There are some basic financial concepts that are quite helpful in understanding term structure modelling. Four basic elements in particular appear consistently in the construction of yield curves: zero-coupon rates, discount factors, par yields, and forward interest rates. The derivation of one of these elements is, conveniently, sufficient for the determination of the other three elements. This section attempts to make clear the links between these elements.

A.1 Zero-coupon rate and discount factors

Each interest rate or bond yield definition is derived from specific representations of the bond price function. If a bond corresponds to a single principal payment to be received at maturity date *T* (i.e., it does not pay a coupon), its price function can be defined in terms of the zero-coupon interest rate *Z(t,T)* for that specific maturity *(T - t)* as follows:[2]

$$Price(t, T) = \frac{100}{(1 + Z(t, T))^{n}} \text{ where } n = (T - t)/365 \text{ in Canada.} \qquad \text{(A:EQ 1)}$$

The zero-coupon interest rate *Z(t,T)* is the yield implied by the difference between a zero-coupon bond’s current purchase price and the value it pays at maturity. A given zero-coupon rate applies only to a single point in the future and, as such, can only be used to discount cash flows occurring on this date. Consequently, there are no embedded assumptions about the investment of intermediate cash flows.

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1. See Fabozzi and Fabozzi (1995) as a supplementary reference.
2. An example of this type of instrument is a Government of Canada treasury bill.

The zero-coupon rate can also be defined in terms of the discount factor for the corresponding term to maturity, which is $Disc(t, T) = (1 + Z(t, T))^{-n}$. The main reason for the usage of the discount factor is its relative ease of use and interpretation in comparison to zero-coupon rates. To calculate a cash flow's present value (or the discounted cash flow), one simply takes the product of this cash flow and the specific discount factor with the corresponding maturity.

The calculation of zero-coupon rates and their related discount factors is particularly relevant for the pricing of coupon bonds. Note that, conceptually, a coupon bond is a portfolio of zero-coupon bonds. A bond with N semi-annual coupon payments (C/2) and a term of maturity of T (or N/2 years) can be priced, using the zero-coupon rates $Z(n)_t$ for each coupon period, from the following relationship:

$$Price(t, T, C) = \sum_{n=1}^{N} \left[\frac{\left(\frac{C}{2}\right)}{(1 + Z(n)_t)^{n/2}} \right] + \frac{100}{(1 + Z(N)_t)^{N/2}}. \qquad \text{(A:EQ 2)}$$

Thus, the price of a bond is simply the sum of its cash flows (coupons and principal) discounted at the zero-coupon interest rates corresponding to each individual cash flow.

Unfortunately, individual zero-coupon rates prevailing in the market are not observable for all maturities. The only Canadian securities from which zero-coupon rates can be extracted directly are treasury bills that have a maximum term to maturity of one year.[3] This implies that zero-coupon rates for longer maturities must be estimated from other securities (i.e., coupon bonds).

A.2 Yield to maturity and the "coupon effect"

For longer maturities, one may observe the prices of Government of Canada bonds, which make semi-annual coupon payments. Bond prices are often summarized by their yield to maturity *(YTM)*, which is calculated as follows:

$$Price(T, C)_t = \sum_{n=1}^{N} \frac{\left(\frac{C}{2}\right)}{(1 + YTM(T, C)_t)^{n/2}} + \frac{100}{(1 + YTM(T, C)_t)^{N/2}}. \qquad \text{(A:EQ 3)}$$

The yield to maturity is the "internal rate of return" or IRR on a bond.[4] That is, it is the constant rate that discounts all the bond's cash flows to obtain the observed price. As a result, the

3. Note that zero-coupon rates for longer maturities could theoretically be observed using Government of Canada bonds that have been stripped into separate coupon and principal components.

4. This calculation is performed with an iterative root-finding algorithm such as Newton-Raphson.

yield to maturity is essentially an average of the various zero-coupon rates, weighted by the timing of their corresponding cash flows. An important, although unrealistic, assumption of the YTM calculation is that all intermediate cash flows are reinvested at the YTM.

The relationship between the YTM for a series of bonds and their term to maturity is frequently used to represent the term structure of interest rates. This is troublesome, given that the size of the coupon will influence the yield-to-maturity measure. In the simplest case of a flat term structure of interest rates, the zero-coupon rate and the yield to maturity will be identical. If $Z(m)_t = Z(n)_t = Z(N)_t = Z_t, \forall((m, n) \in [1, 2, \ldots, N])$, then:

$$(\text{EQ 2}) \Leftrightarrow Price(T, C)_t = \sum_{n=1}^{N}\left[\frac{\left(\frac{C}{2}\right)}{(1 + Z_t)^{n/2}}\right] + \frac{100}{(1 + Z_t)^{N/2}} \qquad \text{(A:EQ 4)}$$

$$(\text{EQ 3}) \Rightarrow Z_t = YTM(T, C)_t. \qquad \text{(A:EQ 5)}$$

Generally, however, the yield curve is not flat and the zero-coupon rates associated with various coupons vary with respect to the timing of the coupon payments.[5] Thus two bonds with identical maturities but different coupons will have different yields to maturity. For example, a larger coupon places a larger weighting on the earlier zero-coupon rates and thus the yield-to-maturity calculation will be different from the lower coupon bond. This is called the "coupon effect."[6] It is particularly problematic in instances where the coupon rate differs substantially from the yield-to-maturity value. This is because the zero-coupon rate weightings are more heavily skewed and the coupon effect is correspondingly larger.

Simply plotting the YTM for a selection of bonds would be misleading. Firstly, the YTM measure, which is a complicated average of zero-coupon rates, cannot be used to discount a single cash flow. In fact, the YTM cannot be used to price any set of bonds apart from the specific bond to which it refers. Secondly, the implicit reinvestment assumption and the coupon effect make the YTM measure extremely difficult to interpret as a yield curve.

5. In a positively (negatively) sloped yield curve environment, the zero-coupon rate for a given maturity will be higher (lower) than the yield to maturity.

6. More accurately, this section describes the "mathematical" coupon effect. It should be noted that differences in the manner in which capital gains and interest income are taxed also gives rise to what is termed the "tax-induced" coupon effect.

A.3 Duration

A concept related to the link between the YTM and prices is the duration of a bond. There are two commonly used measures of duration.[7] The first measure (termed Macauley duration) is a weighted average term to maturity of the present value of the future cash flows of a bond. It is expressed as follows (where *CF* represents cash flow):

$$D(T, C)_t = \frac{\sum_{t=1}^{n} \frac{(CF_t \cdot t)}{(1 + YTM(T, C)_t)^t}}{\sum_{t=1}^{n} \frac{(CF_t)}{(1 + YTM(T, C)_t)^t}}. \qquad \text{(A:EQ 6)}$$

The second measure of duration (termed the modified duration) is a manipulation of the Macauley duration and represents a linear approximation of the convex relationship between the price of a bond and its YTM.[8]

$$D_{Modified} = \frac{D_{(T, C)}}{1 + \left(\frac{YTM}{2}\right)^t}. \qquad \text{(A:EQ 7)}$$

The concept of duration provides a useful method for understanding the relationship between the price and the YTM of a bond. That is, for a given change in a bond's YTM, the change in price will be greater for a longer-term bond than for a shorter-term bond. Duration attempts to quantify this impact. The asymmetry between bond price and yield changes is an important consideration in the modelling of the term structure of interest rates.

A.4 Par yields

To resolve the coupon effect problem in the interpretation of YTM, another representation of the term structure of interest rates called the par yield curve may be used.[9] The par yield for a specific maturity is a theoretical derivate of the YTMs of existing bonds that share this same maturity. It is a YTM that a bond would have if it were priced at par. This means the bond's YTM must be equal to its coupon rate.

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7. See Das (1993a).
8. Convexity implies that changes in yield do not create linear changes in price: As YTM rises, the corresponding price falls at a decreasing rate and, conversely, as YTM falls, the price increases at an increasing rate.
9. The par yield curve and related concepts are well presented in Fettig and Lee (1992).

Since Government of Canada bonds are rarely priced at par in the secondary market, such yields must be estimated from existing bonds' YTMs. It should be noted that a par yield for a single bond cannot be calculated (unless, of course, it is currently trading at par). Instead, a sample of bonds must be used to estimate these hypothetical par yields. Given that the coupon rate is equal to the par yield to maturity ($PAR(t,T)$) and the price by definition is at par (i.e., 100), then the price function of a bond can be rewritten as follows:

$$100 = \sum_{n=1}^{N} \frac{\left(\frac{PAR(t,T)}{2}\right) \cdot 100}{(1 + Z_t(n))^{n/2}} + \frac{100}{(1 + Z_t(N))^{N/2}} \quad . \qquad \text{(A:EQ 8)}$$

A model is required to estimate the par yields that satisfy this equation while simultaneously optimizing the fit with the observed YTMs.[10] A par yield is still a YTM measure. This implies that it has the same characteristics as the YTM: It is a weighted average of zero-coupon rates and assumes all intermediate cash flows are reinvested at the YTM (or par yield).

B. Extracting zero-coupon rates from the par yield curve

One technique used to derive zero-coupon rates from a par yield curve is **"**bootstrapping**."** This technique is a recursive method that divides the theoretical par yield bond into its cash flows and values each independent cash flow as a separate zero-coupon bond.

The method is based on the basic bond pricing formula. By definition, all theoretical par bonds trade with a coupon equal to the YTM and a price equal to $100 (or par). To obtain these par yields, a previously calculated par yield curve is used. The 6-month zero-coupon rate is simply the following, where $PAR(n)_t$ and $Z(n)_t$ are the n-year par yield and zero-coupon rate respectively. In this expression, the 6-month zero-coupon rate is the only unknown variable and can therefore be uniquely determined.

$$100 = \frac{\frac{1}{2}(PAR(0.5)_t \cdot 100) + 100}{(1 + Z(0.5)_t)^{0.5}} \qquad \text{(A:EQ 9)}$$

10. The Super-Bell model is an example of an approach to estimate par yields.

Given the 6-month zero-coupon rate, one may proceed to determine the 1-year rate as follows where $Z(0.5)_t$ is known and one solves for $Z(1)_t$.

$$100 = \frac{\frac{1}{2}(PAR(1)_t \cdot 100)}{(1 + Z(0.5)_t)^{0.5}} + \frac{\frac{1}{2}(PAR(1)_t \cdot 100) + 100}{(1 + Z(1)_t)^{1}} . \qquad \text{(A:EQ 10)}$$

As indicated, this method iterates through each subsequent maturity until zero-coupon values are determined for term to maturities from 0.5 to 30 years. The following box provides a 4-period numerical example for annual zero-coupon rates.[11]

Diagram 1: "Bootstrapping" of zero-coupon rates

Inputs to "Bootstrapping"

par (1) = 5%	par (2) = 6%	par (3) = 7%	par (4) = 8%

Properties of par bonds

time = 1	time = 2	time = 3	time = 4
A **1-period** par bond has a coupon of **5%**.	A **2-period** par bond has a coupon of **6%**.	A **3-period** par bond has a coupon of **7%**.	A **4-period** par bond has a coupon of **8%**.

The 1-period zero-coupon rate is equivalent to the 1-period par yield from the following expression:

$$100 = \frac{105}{(1 + Z(1))^{1}}$$, therefore $Z(1) = 5.00\%$

Using the price function of each individual par bond (1 to *n* periods), it is possible to determine the subsequent zero-coupon rates. To reduce this price function to a single unknown (the zero-coupon rate desired), the previously calculated zero coupon rates are used.

$$100 = \frac{6}{(1.05)^{1}} + \frac{100 + 6}{(1 + Z(2))^{2}}$$, where $Z(2) = 6.03\%$

$$100 = \frac{7}{(1.05)^{1}} + \frac{7}{(1.0603)^{2}} + \frac{100 + 7}{(1 + Z(3))^{3}}$$, where $Z(3) = 7.10\%$

$$100 = \frac{8}{(1.05)^{1}} + \frac{8}{(1.0603)^{2}} + \frac{8}{(1.0710)^{3}} + \frac{100 + 8}{(1 + Z(4)))^{4}}$$, where $Z(4) = 8.22\%$

11. See Das (1993b) for a more detailed example of the bootstrapping technique.

It is important to note that the "bootstrapped" zero-coupon curve will have zero-coupon rates for discrete maturities. The intermediate zero-coupon rates are typically determined by linear interpolation between the discrete maturities.

C. Extracting "implied" forward rates from zero-coupon rates

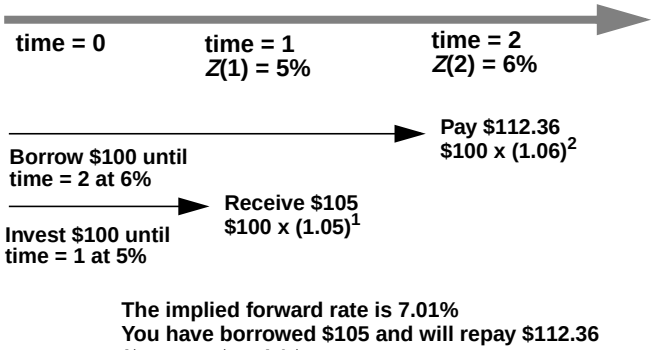
A forward rate is the rate of interest from one period in the future to another period in the future. It is, for example, the rate one would pay (earn) to borrow (lend) money in one year with a maturity in two years. Forward interest rates (like zero-coupon rates) are typically not directly observable and, as a result, they must be derived from the zero-coupon curve; hence the term, "implied" forward rates.

The implied forward rates are derived from zero-coupon rates from an arbitrage argument. Specifically, forward borrowing or lending transactions can be replicated with the appropriate spot transactions. A forward contract, from time τ to T, can be replicated at no cost by borrowing from time t to T and lending the proceeds from t to τ (with $t < \tau < T$). The result is a cash receipt at time τ and an obligation to pay at time T, with the implied rate between period τ and T equal to the forward rate. The following general expression summarizes this argument algebraically:

$$F(t, \tau, T) = \left[\frac{(1 + Z(t, T))^{(T-t)/365}}{(1 + Z(t, \tau))^{(\tau-t)/365}} \right]^{\frac{365}{T-\tau}} - 1 . \qquad \text{(A:EQ 11)}$$

The following box provides a 2-period numerical example of the calculation of implied forward rates:

Diagram 2: Calculation of "implied" forward rates



The implied forward rate is 7.01%
You have borrowed $105 and will repay $112.36
($112.36- $105) / $105 = 0.0701.

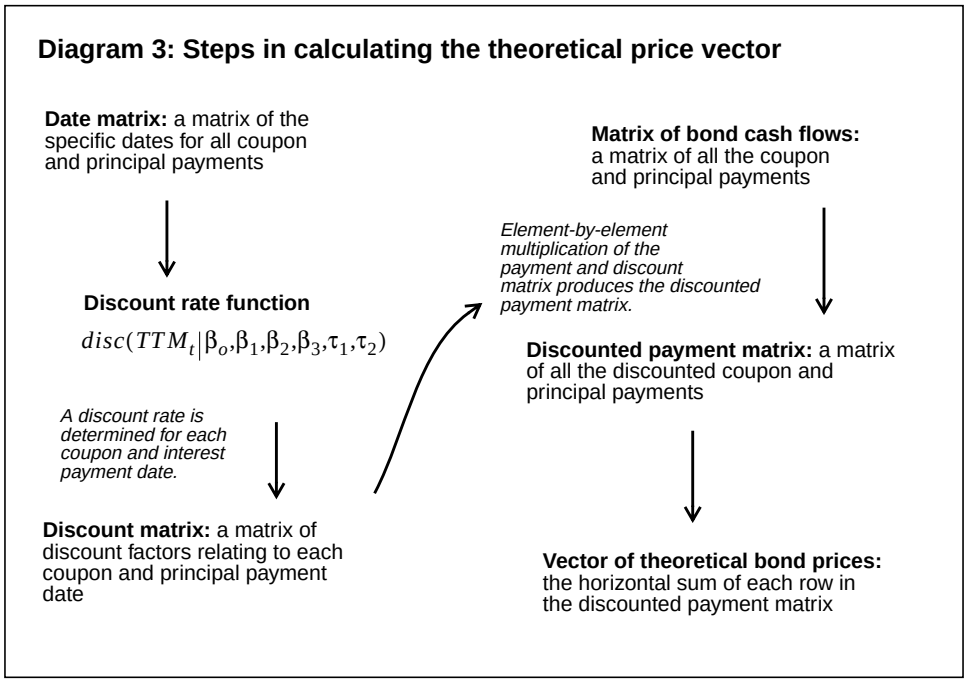
D. Mechanics of the estimation

The estimation of Nelson-Siegel or the Svensson model parameters is briefly described in Figure 2 on page 10 of the text. This appendix provides additional detail on the separate components of the estimation, which can be conceptually divided into three parts: the basic mechanical steps required to generate theoretical bond prices (points C and D of Figure 2); the alternative specifications of the objective function (point F of Figure 2); and finally, the specifics respecting the optimization algorithms (represented in point F of Figure 2). The mechanics of the construction of theoretical bond prices and the alternative formulations of the objective function are described in this section while the optimization strategies are outlined in Section E.

D.1 Construction of theoretical bond prices

The price of a bond is equal to the sum of the discounted values of its cash flows (coupon payments and principal). Therefore, to generate the vector of theoretical bond prices for selected Canadian bonds and bills, a matrix of coupon and principal payments of bonds (matrix of bond cash flows) is built and a matrix of the corresponding coupon and principal payment dates (date matrix) is also constructed. Using the date matrix and the Nelson-Siegel or the Svensson theoretical discount function, a matrix of discount factors (discount matrix) is created relating to the specific interest and coupon payment dates. The Nelson-Siegel and the Svensson discount matrices differ only in the functional form of the forward rate curve functions used for the discount function applied.

The discount matrix and the matrix of bond cash flows are then multiplied in an element-by-element fashion to obtain a matrix of discounted coupon and principal payments (discounted payment matrix). As a final step, the Nelson-Siegel or the Svensson vector of theoretical bond prices is obtained by summing all the discounted payments corresponding to each bond in the discounted payment matrix. This process is outlined in the following diagram. For any set of Svensson parameters, the resulting vector of theoretical bond prices can be calculated.



D.2 Log-likelihood objective function

This name for the objective function is somewhat of a misnomer because maximum likelihood estimation is not actually used. An objective function inspired by a log-likelihood function is derived but there is no particular concern with the distribution of the error terms. What is sought instead is a method that incorporates the information in the bid-offer spread—in particular, the generation of an additional penalty for errors (measured from the mid-price) that fall outside the bid-offer spread.

The vector of bond prices is a function of *X*, described as the matrix of inputs (which includes the cash flow amount and dates), and the vector of term structure parameters. The vector of errors, $\vec{\hat{e}}$, is defined as the difference between the mid-price ($\vec{P}_M = \vec{P}_O + \left(\frac{\vec{P}_B - \vec{P}_O}{2}\right)$) and the estimated price multiplied by a weight matrix (where $diag(\omega)$ is a diagonal matrix with the weight vector as the elements along the main diagonal) as follows:[12]

$$\vec{\hat{e}} = diag(\omega) \cdot [\vec{P}_M - f(X,\vec{\beta})], \text{ where } \vec{\beta} = \{\beta_0 \ldots \beta_n, \tau_1 \ldots \tau_n\} \qquad \text{(A:EQ 12)}$$

12. P_M is defined as the mid-price, P_B as the bid-price, and P_O as the offer-price.

The key assumption of maximum likelihood estimation is the assumption that the errors are normally distributed with a zero mean and a variance of σ^2. This can be expressed as follows:

$$e \sim N(0, \sigma^2 I). \quad \text{(A:EQ 13)}$$

Instead, however, of having a specified constant variance for the errors, there is a unique variance for each observation as one-half of the bid-offer spread or $\overrightarrow{\sigma_{bo}} = \frac{1}{2}(\vec{P}_B - \vec{P}_O)$. This, therefore, transforms the likelihood function into the following:[13]

$$l(\beta, \sigma^2 \big| Price, X) = (2\pi)^{-\frac{N}{2}} |\Omega|^{-\frac{1}{2}} e^{-\frac{(e^T \Omega^{-1} e)}{2}}, \text{ where } \Omega = [diag(\overrightarrow{\sigma_{bo}})]^2 \quad \text{(A:EQ 14)}$$

The final step is to derive the log-likelihood function and apply the appropriate weights. Therefore, in the optimization algorithms, it is desired to maximize the following objective function:

$$L(\beta, \sigma^2 \big| Price, X) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln|\Omega| - \frac{e^T \Omega^{-1} e}{2}. \quad \text{(A:EQ 15)}$$

D.3 Sum of squared errors with penalty parameter objective function

The second objective function used in the estimation of the model is somewhat more straightforward as it represents a special case of simple sum of squared errors objective function. Again, the goal is to penalize to a greater extent those errors that fall outside of the bid-offer spread while maintaining a continuous objective function.

Recall from the previous section (see equation 12) that the vector of bond prices is a function of a matrix of inputs and a vector of parameters. In the simple case, the error is defined in the same way as in the previous section, that is:

$$\vec{\hat{e}} = [\vec{P}_M - f(X, \vec{\beta})] \quad \text{(A:EQ 16)}$$

The transformation is to multiply the price error by two and divide it by the bid-offer spread ($\vec{S} = P_B - P_O$) and raise this quotient to the power of λ, which can be considered as a

13. Note that σ^2 in the likelihood function is set to one. As a result, the likelihood function will differ by a constant from a more general presentation.

penalty parameter (where $diag(\vec{\hat{e}})$ and $diag(\vec{\hat{S}})$ are diagonal matrices with $\vec{\hat{e}}$ and $\vec{\hat{S}}$ as the elements along the main diagonals respectively). This expression is defined as follows:

$$(\psi)^{\lambda}, \text{ where } \psi = 2diag(\vec{\hat{e}}) \cdot diag(\vec{\hat{S}})^{-1}. \qquad \text{(A:EQ 17)}$$

It can be seen that, if the error term in the numerator and the bid-offer spread in the denominator are the same, the value will be one and the exponent will not change its value. The scaling of the error term in the numerator by a factor of two is intended to make this possible. If the error term is less than the bid-offer spread, the value of the expression will be less than one and the exponent will reduce its value, which will correspondingly have less impact in the estimation. Finally, should the error term exceed the bid-offer spread, then the exponent will increase the size of the expression and increase its influence on the estimation. It is worth noting that, in this paper's estimations, the penalty parameter λ was maintained at a fairly arbitrary value of two. It would be possible to increase the penalty for falling outside the bid-offer spread by increasing the size of the penalty parameter. Thus, the sum of squared errors with a penalty parameter objective function is formally expressed as follows:

$$g(\beta|Price,X) = \sqrt{\omega}^{T}(\Psi)^{\lambda}\sqrt{\omega}. \qquad \text{(A:EQ 18)}$$

E. Optimization algorithms

To minimize the weighted sum of the absolute value of price errors, a constrained nonlinear optimization procedure is used. The constraints enhance the speed of the algorithm and avoid "strange" local optima, which are not economically feasible in the Canadian yield curve environment.[14] The specific constraints imposed on the parameter values for both models are as follows:[15]

- Parameters: $0 < \beta_0 < 25 \quad -20 < \beta_1 < 20$, $-25 < \beta_2 < 25$, $-25 < \beta_3 < 25$, $\frac{1}{12} < \tau_1 < 30$, $\frac{1}{12} < \tau_2 < 30$
- Relative values: $0 < \beta_0 + \beta_1$

This is both a discussion and a demonstration of the challenges of determining the parameter values for the parameter models. The two alternative global optimization algorithms that were designed to deal with this problem are discussed in the following sections.

14. These constraints are practically derived and thus do not come from any economic theory. For example, the τs are constrained to the range of Government of Canada bond maturities while the βs are restricted to values that provide reasonable shapes for the resulting zero-coupon and forward curves.

15. The constraints on coefficients β_3 and τ_2, however, only apply to the Svensson model.

E.1 Full-estimation algorithm

This global search algorithm begins with the construction of a matrix of starting parameter values ($S_{i,j}$), runs a local search for each parameter set ($\vec{\beta} = \{\beta_0 \ldots \beta_n, \tau_1 \ldots \tau_n\}$), and then selects the best solution. Conceptually, what is sought is to partition the parameter space and run a local search in each subregion. The dimensionality of the problem, however, makes this practically impossible or, rather, prohibitively time consuming. Therefore, there is an attempt to simplify the grid by making some assumptions about the possible starting values for the β_0 and β_1 parameters. In both the Nelson-Siegel and Svensson models, the β_0 and β_1 parameters are not varied but instead set to "educated guesses" for their values.[16] It is important to note, however, that for each set of starting parameters, the entire parameter set is estimated. The β_0 starting value, which is the asymptote of the instantaneous forward rate function, is set to the YTM of the bond with the longest term to maturity in the data sample (i.e., the most recently issued 30-year bond). It is also noted that, given that the sum of β_0 and β_1 is the vertical intercept of the instantaneous forward rate function, the starting value of β_1 is set to the difference between the longest and shortest YTM in the data set (i.e., the most recently issued 30-year bond YTM less 30-day treasury bill rate).

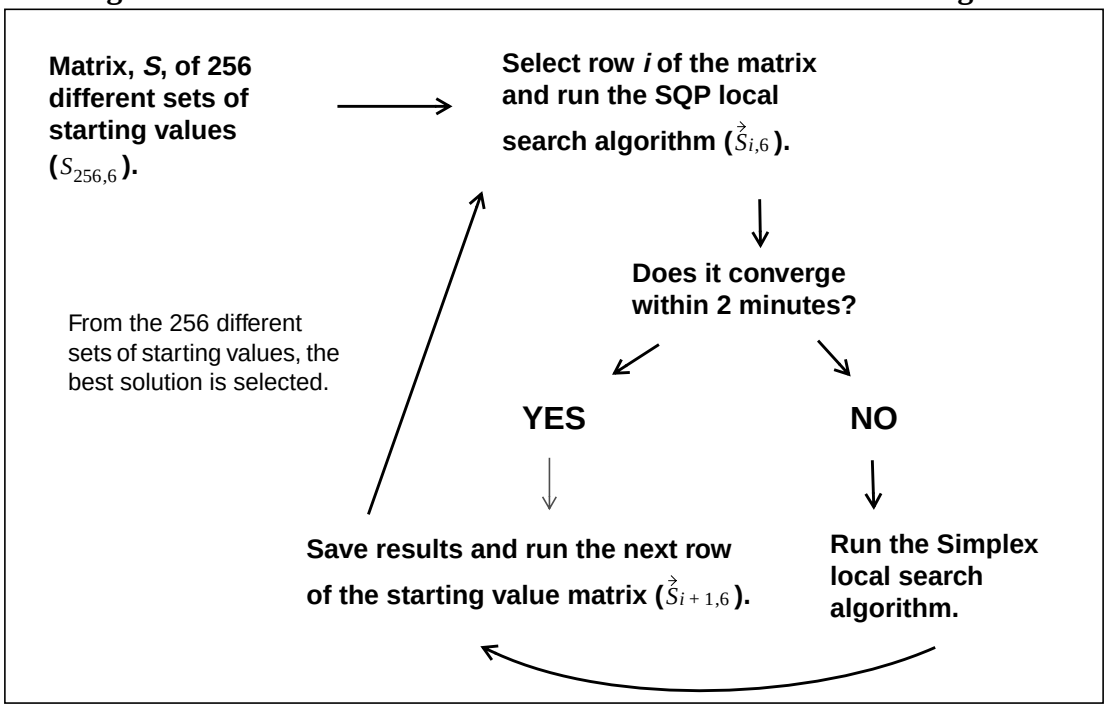
Thus, the previously described values for β_0 and β_1 and combinations of different values for the remaining parameters are used to construct the matrix of starting values. In the Nelson-Siegel model, nine combinations of the remaining two parameters (β_2 and τ_1) are used in the grid for a total of 81 (9^2) sets of starting parameters. In the Svensson model, four combinations of four parameters (β_2, β_3, τ_1, τ_2) are used for a total of 256 (4^4) different sets of starting values. The grid used to estimate the Nelson-Siegel model is much finer than that used for the Svensson model. This is shown by the more robust nature of the results for the Nelson-Siegel model in the text of the paper. The selection of the number of different combinations of starting values appears to be arbitrary although it is really a function of the time constraint. Note that five combinations of the varied parameter set (β_2, β_3, τ_1, τ_2) instead of four combinations for the Svensson model amount to 1,296 (6^4) different sets of starting values. This would require approximately five times longer to estimate than the more-than-three hours already required.

Two alternative local search algorithms are used to solve for each row in the matrix of starting values: sequential quadratic programming (SQP) and the Nelder and Mead Simplex method. Sequential quadratic programming uses local gradient information to determine the

16. Ricart and Sicsic (1995) use these as constraints in their estimation. The ideas are used in this paper without constraining the parameters.

direction of movement of the algorithm over the objective function; the Simplex method uses a series of direct function evaluations to determine the direction of descent.[17] The reason for two alternative approaches is the difficulties in the estimation of the full model. On occasion, the SQP algorithm fails to converge. To solve this problem, it was decided to limit the time permitted for each SQP local search. On failure to converge before a specified period of time (two minutes), it would be replaced with the more reliable although less accurate Simplex algorithm with which there had been no convergence problems.[18] Figure 1 provides a simple flow chart of the steps in the full-estimation algorithm for the Svensson model (the logic of the Nelson-Siegel model is identical).

Figure 1. A flow chart of the "Svensson model" full-estimation algorithm



E.2 Partial-estimation algorithm

This global search algorithm divides the parameters into two groups, the βs (or linear parameters) and the τs (or the non-linear parameters). It works in a number of steps where one group of parameters is fixed while the other is estimated. The advantages of estimating one subset of the parameters while holding the other constant are improved speed of convergence and increased stability. Indeed, unlike the full-estimation algorithm discussed in the previous section,

17. See C. F. Garvin, "Introduction to Algorithms," in *Financial Toolbox: For Use with MATLAB* (Natick, Mass.: MathWorks, 1996) for a good discussion of the SQP algorithm, and Nelder and Mead (1965) for the original description of the surprisingly straightforward Simplex algorithm.

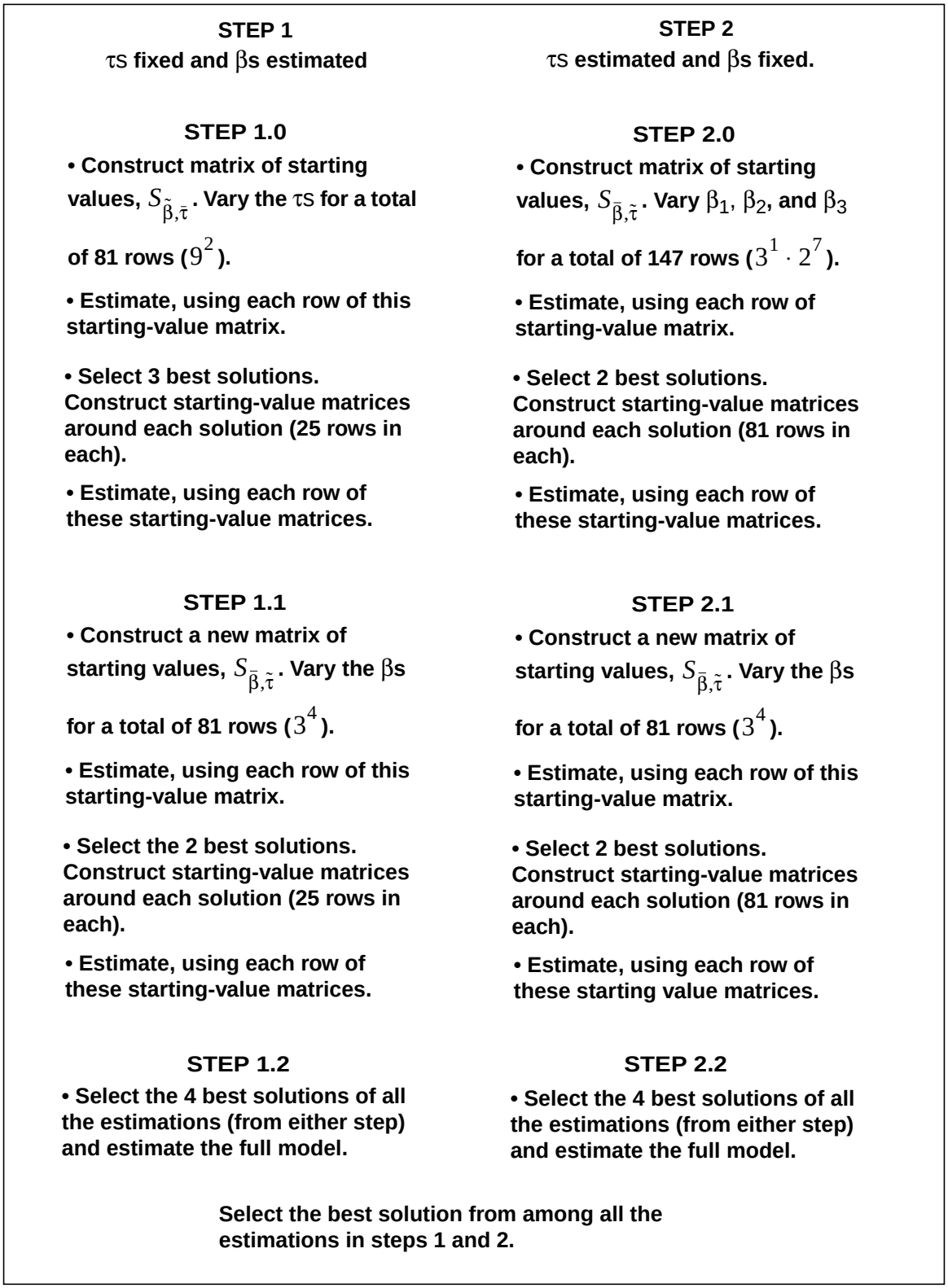
18. Two minutes was chosen as the cut-off for the SQP algorithm through a process of trial and error.

each step of the partial-estimation algorithm was estimated using the SQP algorithm exclusively and no convergence problems were encountered.

Whenever certain sets of parameters are fixed, there is a concern that the solution is being constrained from its optimum. The partial estimations were performed in two separate streams in an attempt to mitigate this concern. The first stream fixes the τ parameters while estimating the βs, and then proceeds to fix the βs and estimate the τs. The second stream proceeds in reverse fashion: It fixes the βs and estimates the τs, and subsequently fixes the τs and estimates the βs. Each step of both streams uses the best solution or, rather, set of estimated parameters from the previous step. Specifically, a new matrix of starting parameters is then built around these estimated parameters to perform a new round of partial estimations. Note that, in each new matrix of starting parameters, only those parameters that are fixed in that round of estimation are varied, while the estimated parameters use the previous step's estimated parameters as starting values. Upon completion of both streams, all the estimations performed (both partial and full) are sorted by the value of their objective function and the best solution is selected. The two main streams (termed Step 1 and Step 2) are outlined in Figure 2.

Note that both estimation streams begin with a relatively coarse matrix of starting parameters; that is, using a wide range of values for both the fixed and the estimated parameters. This allows, in subsequent steps, the estimation of the parameters that were fixed in the first step using a narrower grid. It thereby permits the analysis to be focused around the best estimated values obtained in the first step. The estimation of all the parameters can therefore be performed in the final step for a small number of starting parameters.

Figure 2. A flow chart of "Svensson model" partial-estimation algorithm



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